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DERIVING THE ERROR OF TIME FILTERED LEAPFROG SCHEME VIA MODIFIED EQUATIONS

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ABSTRACT

The leapfrog (LF) scheme is a cornerstone of numerical weather prediction and large-scale atmospheric modeling due to its computational efficiency and ability to preserve the amplitude of pure oscillations during long integrations. However, the three-time-level nature of the LF method introduces a parasitic computational mode that can grow over time and contaminate physical solutions. Traditionally, the Robert-Asselin (RA) filter has been employed to suppress this mode, but it inadvertently damps the physical mode, reducing the LF scheme's formal accuracy from second to first order. This research provides a rigorous mathematical analysis of modern time filters—specifically the Robert Asselin (RA), Robert Asselin Williams (RAW), and higher-order Robert Asselin (hoRA) filters— using the method of modified equations to evaluate phase and amplitude errors. By solving the linear system for each filtered scheme, we derive equivalent linear multistep methods and their corresponding two-term modified equations. Our findings confirm that the RAW filter significantly mitigates the physical mode damping of the RA filter, recovering second-order accuracy when parameters are optimally tuned (e.g., $\alpha = 0.53$). Furthermore, the hoRA filter demonstrates even higher performance, attaining second-order accuracy generally and third-order accuracy for the specific choice of $\beta = 0.4$. Numerical tests on the oscillation equation validate these theoretical derivations, showing that the hoRA filter yields the lowest amplitude and phase error magnitudes compared to the RA and RAW alternatives.

1. INTRODUCTION

The leapfrog (LF) scheme has long been a preferred time-stepping method in numerical weather prediction because it is both accurate and efficient. It offers a major advantage in its ability to preserve the amplitude of pure oscillations, avoiding the excessive numerical damping common in other schemes—an important property for long integrations. Additionally, the LF method is computationally efficient, requiring only one evaluation of the model tendencies per time step, making it attractive for large-scale atmospheric models. The leapfrog method, which employs data from three distinct time points, generates two distinct solution modes when applied to linear systems: a physical mode that represents the true solution and a computational mode that is purely numerical. While the computational mode remains neutrally stable in linear problems, nonlinearities can couple it with the physical mode and cause its amplitude to grow. Although this growth is usually negligible in short simulations, it can become significant in long integrations, eventually allowing the computational mode to contaminate or even dominate the solution. The Robert Asselin (RA) filter has been widely studied, particularly with respect to its accuracy and stability [1-8, 20,21] which finds extensive utility in operational weather forecasting, global climate simulations, ocean dynamics modeling, and specialized rotating-annulus studies. Despite its popularity, the RA filter has a significant drawback: its effect extends beyond the computational mode, also damping the physical mode. This unwanted damping reduces the leapfrog scheme's formal accuracy from second to first order and can degrade simulation quality. Consequently, physical invariants such as energy that are conserved in the continuous equations may no longer be conserved once the RA filter is applied. Since many atmospheric models rely on legacy code built around the RA-filtered leapfrog scheme, there is strong interest in simple, non-intrusive improvements that avoid major code redesign. In this context, Williams [7, 9,10] introduced a modified version of the RA filter designed to enhance performance while remaining compatible with the existing leapfrog framework. This modification, known as the Robert–Asselin–Williams (RAW) filter, greatly mitigates the RA filter's excessive damping of the physical mode while recovering the leapfrog scheme's second-order accuracy. This improvement comes with a minor loss of stability, but the filter has nonetheless been widely implemented and investigated in [8, 10-12]. According to Li and Trenchea [13], the higher-order Robert–Asselin (hoRA) time filter attains third-order accuracy while maintaining the computational efficiency of the RAW filter. When applied to the leapfrog method, the hoRA filter yields performance— in terms of accuracy, stability, and efficiency— comparable to the third-order Adams–Bashforth method [14, 15], yet remains easy to implement within existing legacy codes. To assess how different time filters influence the phase and amplitude characteristics of the leapfrog scheme, we conduct a modified-equation analysis, which also offers insight useful for evaluating errors in higher-order linear multistep methods.

2. MATERIALS AND METHODS

2.1. Preliminaries

The phase and amplitude accuracy of time-integration methods applied to non-dissipative dynamical systems is often modeled and analyzed by solving the generalized oscillation equation (see [16, 17])

$$u'(t) = i\omega u(t) \text{ and } u(0) = 1 \tag{1}$$

Here, i signifies the imaginary unit, and ω is a real-number constant. we summarize the main properties of the RA, RAW, and hoRA time filters. When applied to equation (1), the leapfrog scheme incorporating the RAW filter takes the form.

$$\begin{aligned}
u_{n+1}^{**} &= u_{n-1} + 2i\omega\Delta t u_n^{*}, && (LF)\\
u_n &= u_n^{*} + \frac{\alpha\nu}{2}\left(u_{n+1}^{**} - 2u_n^{*} + u_{n-1}\right), && (RA)\\
u_{n+1}^{*} &= u_{n+1}^{**} + \frac{(\alpha-1)\nu}{2}\left(u_{n+1}^{**} - 2u_n^{*} + u_{n-1}\right), && (W)
\end{aligned}$$

where the symbols u^{**}, u^{*} and u indicate the unfiltered solution, the filtered solution, and the solution after two filter applications, respectively. These parameters, expressed in nondimensional form, satisfy $\nu \in [0,1]$ and $\alpha \in [0.5,1]$. When $\alpha = 1$ the (W) term vanishes, recovering the LF-RA scheme; similarly, setting $\nu = 0$ yields the standard leapfrog (LF) method. The RA and RAW filters are first-order accurate and suppress the computational mode, with the RAW filter better preserving the physical-mode amplitude. If α=0.53 the LF–RAW formulation conserves the mean across the three-time levels and achieves second-order accuracy, while its representation of the physical-mode amplitude reaches third-order accuracy. Despite this advantage, the LF–RAW scheme becomes unconditionally unstable at this value of α. When α is chosen slightly above 0.5—for example, α=0.53 the LF-RAW scheme still produces physical-mode amplitudes that are nearly third-order accurate (see [7, 18]). The leapfrog scheme with the hoRA filter (LF-hoRA), when applied to equation (1), is expressed as follows

$$\begin{aligned}
u_{n+1}^{*} &= u_{n-1} + 2i\omega\Delta t u_n^{*} && (LF)\\
u_n &= u_n^{*} + \frac{\beta}{2}\left(u_{n+1}^{*} - 2u_n^{*} + u_{n-1}\right) - \frac{\beta}{2}\left(u_n^{*} - 2u_{n-1} + u_{n-2}\right) && (hoRA)
\end{aligned}$$

where u and u^{*} are correspond to the filtered and unfiltered solutions in the same order, and β∈[0,1] denotes a nondimensional filter parameter. In the asymptotic limit of adequate temporal resolution, ωΔt≪1, the LF-hoRA scheme attains second order accuracy in general, while third order accuracy is achieved for the specific choice β=0.4. (see [13]).

2.2. Linear Multistep Methods of LF-RA, LF-RAW and LF-hoRA

We first solve the linear system (LF)-(RA-(W) for u_n^{*} and u_{n+1}^{*} in terms of u_n, u_{n-1} we acquired

$$u_n^* = \frac{u_n - \nu\alpha u_{n-1}}{1 + \nu\alpha i\omega h - \nu\alpha},$$
$$u_{n+1}^* = (1 - \nu + \nu\alpha)u_{n-1} + (2i\omega h - \nu i\omega h + \nu + \nu\alpha i\omega h - \nu\alpha)\frac{u_n - \nu\alpha u_{n-1}}{1 + \nu\alpha i\omega h - \nu\alpha}$$

By comparing the expression for u_{n+1}^* with the corresponding expression for u_n^* after applying the index shift from $n\ to\ n+1$, we find that the filtered leapfrog with RAW scheme can be written in the form of an equivalent following linear multistep method,

$$u_{n+1} \quad = \nu u_n + (1-\nu)u_{n-1} + i\omega h(2 + \nu\alpha - \nu)u_n - i\omega h\nu\alpha u_{n-1}.$$

Moreover, setting $\alpha = 1$ in this formulation recovers the standard RA-filtered leapfrog scheme. Similar to (LF)-(RA)-(W) linear multistep method, The linear system is solved with the leapfrog method combined with the hoRA filter (LF-hoRA) for u_n^* and u_{n+1}^* in terms of $u_n,\ u_{n-1}$ and u_{n-2}, we obtain

$$u_n^* = \frac{u_n - 2\beta u_{n-1} + \frac{\beta}{2}u_{n-2}}{1 - \frac{3\beta}{2} + i\omega\Delta t\beta},$$
$$u_{n+1}^* = u_{n-1} + \frac{4i\omega\Delta t + 2i\omega\Delta t\beta - 2i\omega\Delta t\beta}{2 - 3\beta + 2i\omega\Delta t\beta}\left(u_n - 2\beta u_{n-1} + \frac{\beta}{2}u_{n-2}\right).$$

By matching the expression for u_{n+1}^* with the corresponding expression u_n^* after replacing indices from $n\ to\ n+1$, we conclude that the hoRA-filtered leapfrog scheme is equivalent to the following linear multistep method

$$u_{n+1} = 2\beta u_n + (1-2\beta)u_{n-1} + i\omega\Delta t(2u_n - 3\beta u_{n-1} + \beta u_{n-2}).$$

In this work we focus on a new approach employs modified equations to evaluate the phase and amplitude errors of time filters commonly applied to the leapfrog scheme, namely, LF-RA, LF-RAW and LF-hoRA.

3. RESULTS AND DISCUSSION

3.1. Error Analysis with Modified Equations

We will analyze the behavior of time filtered leapfrog scheme on the oscillation equation (1) by using two term modified equation [19]. We denote the phase and amplitude error as R_+ and A_+ respectively.

3.1.1. Leapfrog with Robert Asselin Time Filter

Theorem 1: The phase and amplitude error in LF Robert-Asselin (RA) time filter applied to oscillation equation (1) is

$$R_+ = \frac{1+\nu}{3(2-\nu)}\omega^2h^2 + \mathcal{O}(h^4)$$
$$|A_+| = -\frac{\nu}{2(2-\nu)}\omega^2h^2 + \mathcal{O}(h^4).$$

Proof: Consider linear multistep method of LF-RA time filter with oscillation equation (1),

$$u_{n+1} \quad = \nu u_n + (1-\nu)u_{n-1} + 2i\omega h u_n - i\omega h\nu u_{n-1} \tag{2}$$

The LF method constitutes a second-order approximation to the oscillation equation, in contrast to LF-RA, which is first-order accurate. The associated two-term modified equation for (1) is

$$y' = i\omega y + hg_1(y) + h^2g_2(y)$$

Thus,

$$y'' = -\omega^2y + i\omega hg_1(y) + i\omega hg_1'(y)y + \mathcal{O}(h^2)$$
$$y''' = -i\omega^3y + \mathcal{O}(h).$$

Local truncation error of LF-RA applied modified equation with assumption of all previous numerical solutions are exact i.e. $u_n = u(t_n), u_{n-1} = u(t_{n-1}), \ldots, u_1 = u(t_1)$ is

$$\begin{aligned} h\tau_n &= y(t_{n+1}) - y_{n+1} \\ &= y_{n+1} - \nu y_n - (1-\nu)y_{n-1} - 2i\omega h y_n + i\omega h\nu y_{n-1} \end{aligned} \tag{3}$$

Apply Taylor expansion of y_{n-1} and y_{n+1} at time t_n and substitute in (3),

$$\begin{aligned} h\tau_n &= \left(y_n + hy_n' + \frac{h^2}{2}y_n'' + \frac{h^3}{6}y_n''' + O(h^4) \right) - \nu y_n \\ &\quad -(1-\nu)(y_n - hy_n' + \frac{h^2}{2}y_n'' - \frac{h^3}{6}y_n''' + O(h^4)) - 2i\omega h y_n \\ &\quad + i\omega h\nu \left(y_n - hy_n' + \frac{h^2}{2}y_n'' - \frac{h^3}{6}y_n''' + O(h^4) \right) \end{aligned}$$

Eliminate higher order terms and rearrange,

$$\begin{aligned} h\tau_n &= (1 - \nu - 1 + \nu - 2i\omega h + i\omega h\nu)y_n + (1 + 1 - \nu - i\omega h\nu)hy_n' \\ &\quad +(1 - 1 + \nu + i\omega h\nu)\frac{h^2}{2}y_n'' + (1 + 1 - \nu - i\omega h\nu)\frac{h^3}{6}y_n''' + O(h^4) \\ &= (-2i\omega h + i\omega h\nu)y_n + (2 - \nu - i\omega h\nu)hy_n' \\ &\quad +(\nu + i\omega h\nu)\frac{h^2}{2}y_n'' + (2-\nu)\frac{h^3}{6}y_n''' + O(h^4) \end{aligned}$$

Now substitute y_n', y_n'' and y_n''' and simplify,

$$\begin{aligned} h\tau_n &= \left(\frac{\omega^2\nu}{2}y_n + (2-\nu)g_1(y_n) \right) h^2 \\ &\quad +(-\frac{i\omega\nu}{2}g_1(y_n) + \frac{i\omega\nu}{2}g_1'(y_n)y_n - \frac{i\omega^3}{3}y_n - \frac{i\omega^3\nu}{3}y_n + (2-\nu)g_2(y_n))h^3. \end{aligned}$$

set coefficient of h^2 term to zero, we get $g_1(y_n)$

$$\frac{\omega^2\nu}{2}y_n + (2-\nu)g_1(y_n) = 0 \Longrightarrow g_1(y_n) = -\frac{\nu}{2(2-\nu)}\omega^2 y_n$$

Now substitute $g_1(y_n)$ and $g_1(y_n)'$in the coefficient of h^2 term then set it to zero, we find $g_2(y_n)$,

$$-\frac{i\omega\nu}{2}(-\frac{\nu}{2(2-\nu)}\omega^2 y_n) + \frac{i\omega\nu}{2}(-\frac{\nu}{2(2-\nu)}\omega^2)y_n - \frac{i\omega^3}{3}y_n - \frac{i\omega^3\nu}{3}y_n + (2-\nu)g_2(y_n) = 0$$

$$\Longrightarrow g_2(y_n) = \frac{1+\nu}{3(2-\nu)}i\omega^3 y_n.$$

Thus, the two term modified equation corresponding to the oscillation equation for LF-RA is

$$y' = i\omega y - \frac{\nu}{2(2-\nu)}\omega^2 h y + \frac{1+\nu}{3(2-\nu)}i\omega^3 h^2 y$$

The analytical solution of the oscillation equation (u_{sol}) and two term modified equation (y_{sol}) for RA is

$$u_{sol} = e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$$

$$\begin{aligned} y_{sol} &= e^{(i\omega - \frac{\nu}{2(2-\nu)}\omega^2 h + \frac{1+\nu}{3(2-\nu)}i\omega^3 h^2)t} \\ &= e^{(i\omega + \frac{1+\nu}{3(2-\nu)}i\omega^3 h^2)t} e^{(-\frac{\nu}{2(2-\nu)}\omega^2 h)t} \end{aligned}$$

$$= e^{(-\frac{\nu}{2(2-\nu)}\omega^2 h)t}(\cos(\omega t + \tfrac{1+\nu}{3(2-\nu)}\omega^3 h^2 t) + i\sin(\omega t + \tfrac{1+\nu}{3(2-\nu)}\omega^3 h^2 t)).$$

For the LF-RA method, the physical-mode phase speed error is given by

$$R_+ - 1 = \frac{arg(y_{sol})}{arg(u_{sol})} - 1 = \frac{\omega t + \frac{1+\nu}{3(2-\nu)}\omega^3 h^2 t}{\omega t} - 1 = \frac{1+\nu}{3(2-\nu)}\omega^2 h^2 + \mathcal{O}(h^3).$$

The amplitude of the physical mode in the LF-RA method is determined to be

$$\begin{aligned} |A_+| = |y_{sol}| - |u_{sol}| \quad &= \left| e^{\left(i\omega + \frac{1+\nu}{3(2-\nu)} i\omega^3 h^2\right)t} e^{\left(-\frac{\nu}{2(2-\nu)}\omega^2 h\right)t} \right| - 1 \\ &= \left| e^{\left(-\frac{\nu}{2(2-\nu)}\omega^2 h\right)t} \right| - 1 + \mathcal{O}(h^4). \end{aligned}$$

As we are examining the local error, we set $t = h$ and consider $\boldsymbol{x}$ to be very small, $e^\wedge x \approx 1 + x$ then

$$|A_+| = |y_{sol}| - |u_{sol}| = 1 - \tfrac{\nu}{2(2-\nu)}\omega^2 h^2 - 1 + \mathcal{O}(h^3) = -\tfrac{\nu}{2(2-\nu)}\omega^2 h^2 + \mathcal{O}(h^3).$$

3.1.2. The RAW Filtered Leapfrog Scheme

Theorem 2: The phase and amplitude error in Robert-Asselin-Williams (RAW) time filter applied to oscillation equation (1) is

$$R_+ = -\frac{3\nu^2(\alpha-1)(2\alpha-1) + (\nu - 3\nu\alpha - 2)(2-\nu)}{6(2-\nu)^2} h^2\omega^2 + \mathcal{O}(h^4),$$

$$|A_+| = \frac{(2\nu\alpha - \nu)}{2(2-\nu)} h^2\omega^2 + \mathcal{O}(h^4).$$

Proof: Consider linear multistep method of LF-RAW time filter applied to oscillation equation (1),

$$u_{n+1} = \nu u_n + (1-\nu)u_{n-1} + i\omega h(2 + \nu\alpha - \nu)u_n - i\omega h\nu\alpha u_{n-1} \tag{4}$$

Recall that the leapfrog (LF) method provides a second-order approximation to the oscillation equation, whereas the LF-RAW scheme yields only a first-order approximation. The two-term modified equation corresponding to the oscillation equation for LF-RAW is given by

$$y' = i\omega y + hg_1(y) + h^2 g_2(y).$$

Therefore,

$$\begin{aligned} y'' &= -\omega^2 y + i\omega h g_1(y) + i\omega h g_1'(y)y + \mathcal{O}(h^2). \\ y''' &= -i\omega^3 y + \mathcal{O}(h). \end{aligned}$$

Local truncation error of LF-RAW applied modified equation with assumption of all previous numerical solutions are exact i.e. $u_n = u(t_n), u_{n-1} = u(t_{n-1}), \dots, u_1 = u(t_1)$ is

$$\begin{aligned} h\tau_n &= y(t_{n+1}) - y_{n+1} \\ &= y(t_{n+1}) - (\nu y_n + (1-\nu)y_{n-1} + i\omega h(2 + \nu\alpha - \nu)y_n - i\omega h\nu\alpha y_{n-1}) \\ &= (-2i\omega h + i\omega h\nu + 2i\omega h - i\omega h\nu)y_n \\ &\quad + ((\omega^2\nu\alpha - \frac{\omega^2\nu}{2})y_n + (2-\nu)g_1(y_n))h^2 + (\frac{i\omega\nu}{2}g_1(y_n) - i\omega\nu\alpha g_1(y_n) \\ &\quad + \frac{i\omega\nu}{2}g_1'(y_n)y_n - \frac{i\omega^3\nu\alpha}{2}y_n - \frac{i\omega^3}{3}y_n + \frac{i\omega^3\nu}{6}y_n + (2-\nu)g_2(y_n))h^3 + \mathcal{O}(h^4). \end{aligned}$$

Set coefficient of h^2 term to zero, we find $g_1(y)$,

$$\left(\omega^2\nu\alpha - \frac{\omega^2\nu}{2}\right)y_n + (2-\nu)g_1(y_n) = 0 \Longrightarrow g_1(y_n) = -\frac{(2\nu\alpha - \nu)}{2(2-\nu)}\omega^2 y_n$$

Now substitute $g_1(y_n)$ and $g_1(y_n)'$ in the coefficient of h^3 term, then set it to zero, we find $g_2(y_n)$,

$$[-i\omega\nu\alpha(-\frac{(2\nu\alpha-\nu)}{2(2-\nu)}\omega^2 y_n)+\frac{i\omega\nu}{2}\left(-\frac{(2\nu\alpha-\nu)}{2(2-\nu)}\omega^2 y_n\right)$$
$$+\frac{i\omega\nu}{2}(-\frac{(2\nu\alpha-\nu)}{2(2-\nu)}\omega^2)y_n-\frac{i\omega^3\nu\alpha}{2}y_n-\frac{i\omega^3}{3}y_n+\frac{i\omega^3\nu}{6}y_n+(2-\nu)g_2(y_n)]=0$$
$$\Rightarrow g_2(y_n)=-\frac{3\nu^2(\alpha-1)(2\alpha-1)+(\nu-3\nu\alpha-2)(2-\nu)}{6(2-\nu)^2}i\omega^3 y_n.$$

Thus, the two term form of the modified oscillation equation for LF-RAW is

$$y'=i\omega y-\frac{(2\nu\alpha-\nu)}{2(2-\nu)}h\omega^2 y-\frac{3\nu^2(\alpha-1)(2\alpha-1)+(\nu-3\nu\alpha-2)(2-\nu)}{6(2-\nu)^2}h^2 i\omega^3 y$$

The analytical solution of the oscillation and two term modified equation for LF-RAW is

$$u_{sol}\quad = e^{i\omega t}=\cos(\omega t)+i\sin(\omega t).$$
$$y_{sol}\quad = e^{i\omega t-\frac{(2\nu\alpha-\nu)}{2(2-\nu)}h\omega^2 t-\frac{3\nu^2(\alpha-1)(2\alpha-1)+(\nu-3\nu\alpha-2)(2-\nu)}{6(2-\nu)^2}h^2 i\omega^3 t}$$
$$= e^{-\frac{(2\nu\alpha-\nu)}{2(2-\nu)}h\omega^2 t}(\cos(\omega t-\frac{3\nu^2(\alpha-1)(2\alpha-1)+(\nu-3\nu\alpha-2)(2-\nu)}{6(2-\nu)^2}h^2\omega^3 t)$$
$$+i\sin(\omega t-\frac{3\nu^2(\alpha-1)(2\alpha-1)+(\nu-3\nu\alpha-2)(2-\nu)}{6(2-\nu)^2}h^2\omega^3 t)).$$

The LF-RAW scheme exhibits the following phase speed error for the physical mode

$$R_+-1\quad =\frac{arg(y_{sol})}{arg(u_{sol})}-1=\frac{\omega t-\frac{3\nu^2(\alpha-1)(2\alpha-1)+(\nu-3\nu\alpha-2)(2-\nu)}{6(2-\nu)^2}h^2\omega^3 t}{\omega t}-1$$
$$=-\frac{3\nu^2(\alpha-1)(2\alpha-1)+(\nu-3\nu\alpha-2)(2-\nu)}{6(2-\nu)^2}h^2\omega^2+\mathcal{O}(h^4)$$

For the LF-RAW method, the physical-mode amplitude is given by

$$|A_+|=|y_{sol}|-|u_{sol}|\quad =\left|e^{i\omega t-\frac{(2\nu\alpha-\nu)}{2(2-\nu)}h\omega^2 t-\frac{3\nu^2(\alpha-1)(2\alpha-1)+(\nu-3\nu\alpha-2)(2-\nu)}{6(2-\nu)^2}h^2 i\omega^3 t}\right|-1$$
$$=\left|e^{-\frac{(2\nu\alpha-\nu)}{2(2-\nu)}h\omega^2 t}\right|-1+\mathcal{O}(h^4).$$

To analyze the local error, we let $t=h$ and consider x in the limit of small values, $e^\wedge x\ \approx 1+x$ then

$$|A_+|=\ |y_{sol}|-|u_{sol}|\quad =1-\frac{(2\nu\alpha-\nu)}{2(2-\nu)}h^2\omega^2-1+\mathcal{O}(h^3)=\frac{(2\nu\alpha-\nu)}{2(2-\nu)}h^2\omega^2+\mathcal{O}(h^4).$$

3.1.3. Leapfrog Method Enhanced with Higher-Order Robert–Asselin (hoRA) Filter

Theorem 3: Phase and amplitude errors associated with the higher-order Robert–Asselin (hoRA) time filter. applied to oscillation equation (1) is

$$R_+=\frac{2-5\beta}{12(1-\beta)}\omega^2 h^2+\mathcal{O}(h^4)$$

$$|A_+|=\frac{\beta(2\beta-3)}{8(1-\beta)^2}\omega^4 h^4+\mathcal{O}(h^6)$$

Proof: Consider linear multistep method of LF-hoRA with oscillation equation (1),

$$u_{n+1} = 2\beta u_n + (1-2\beta)u_{n-1} + 2i\omega h u_n - 3i\omega h\beta u_{n-1} + i\omega h\beta u_{n-2} \tag{5}$$

Recall that leapfrog (LF) method provides a second-order approximation of the oscillation equation, and the LF-hoRA scheme retains this second-order accuracy while incorporating the higher-order Robert–Asselin time filter. Two term modified equation of oscillation equation is

$$y' = i\omega y + h^2 g_1(y) + h^3 g_2(y).$$

Thus,

$$\begin{aligned} y'' &= -\omega^2 y + i\omega h^2 g_1(y) + i\omega h^2 g_1'(y)y + \mathcal{O}(h^3). \\ y''' &= -i\omega^3 y + \mathcal{O}(h^2). \\ y^{(4)} &= w^4 y + \mathcal{O}(h^2). \end{aligned}$$

Local truncation error of LF-hoRA applied modified equation with assumption of all previous numerical solutions are exact i.e. $u_n = u(t_n), u_{n-1} = u(t_{n-1}), \dots, u_1 = u(t_1)$ is

$$\begin{aligned} h\tau_n &= y(t_{n+1}) - y_{n+1} \\ &= y_{n+1} - (2\beta y_n + (1-2\beta)y_{n-1} + 2i\omega h y_n - 3i\omega h\beta y_{n-1} + i\omega h\beta y_{n-2}) \\ &= \left(\left(\frac{i\omega^3\beta}{2} - \frac{i\omega^3}{3} + \frac{i\beta\omega^3}{3}\right)y_n + (2-2\beta)g_1(y_n)\right)h^3 \\ &\quad +((\frac{5\omega^4\beta}{6} + \frac{\omega^4\beta}{12})y_n - i\omega\beta g_1(y_n) + (2-2\beta)g_2(y_n) + i\omega\beta g_1(y_n) \\ &\qquad + i\omega\beta g_1'(y_n)y_n)h^4 + \mathcal{O}(h^5). \end{aligned}$$

Set coefficient of h^3 term to zero, we obtain $g_1(y)$

$$\left(\frac{i\omega^3\beta}{2} - \frac{i\omega^3}{3} + \frac{i\omega^3\beta}{3}\right)y_n + (2-2\beta)g_1(y_n) = 0 \Longrightarrow g_1(y_n) = \frac{2-5\beta}{12(1-\beta)}i\omega^3 y_n$$

Replace $g_1(y)$ and $g_1(y)'$ in the coefficient of h^4 term then set it to zero, we obtain $g_2(y_n)$,

$$\begin{aligned} &(\frac{5\omega^4\beta}{6} + \frac{\omega^4\beta}{12})y_n + (2-2\beta)g_2(y_n) + i\omega\beta\frac{2-5\beta}{12(1-\beta)}i\omega^3 y_n = 0 \\ &\Longrightarrow g_2(y_n) = \frac{6\beta^2 - 9\beta}{24(1-\beta)^2}\omega^4 y_n = \frac{\beta(2\beta-3)}{8(1-\beta)^2}\omega^4 y_n \end{aligned}$$

Thus, the oscillation equation expressed as a two-term modified equation for LF-hoRA is

$$y' = i\omega y + \frac{2-5\beta}{12(1-\beta)}i\omega^3 h^2 y + \frac{\beta(2\beta-3)}{8(1-\beta)^2}\omega^4 h^3 y$$

The analytical solution of the oscillation and two term modified equation for LF-hoRA is

$$\begin{aligned} u_{sol} &= e^{i\omega t} = \cos(\omega t) + i\sin(\omega t) \\ y_{sol} &= e^{i\omega t + \frac{2-5\beta}{12(1-\beta)}i\omega^3 h^2 t + \frac{\beta(2\beta-3)}{8(1-\beta)^2}\omega^4 h^3 t} \\ &= e^{\frac{\beta(2\beta-3)}{8(1-\beta)^2}\omega^4 h^3 t}(\cos(\omega t + \frac{2-5\beta}{12(1-\beta)}\omega^3 h^2 t) + i\sin(\omega t + \frac{2-5\beta}{12(1-\beta)}\omega^3 h^2 t)). \end{aligned}$$

The phase speed error associated with the physical mode of the LF-hoRA method can be expressed as

$$R_+ - 1 = \frac{arg(y_{sol})}{arg(u_{sol})} - 1 = \frac{\omega t + \frac{2-5\beta}{12(1-\beta)}\omega^3 h^2 t}{\omega t} - 1 = \frac{2-5\beta}{12(1-\beta)}\omega^2 h^2 + \mathcal{O}(h^4).$$

The LF-hoRA scheme yields the following amplitude for the physical mode

$$\begin{aligned}|y_{sol}|-|u_{sol}| &= \left|e^{i\omega t+\frac{2-5\beta}{12(1-\beta)}i\omega^3h^2t+\frac{\beta(2\beta-3)}{8(1-\beta)^2}\omega^4h^3t}\right|-1\\ &= \left|e^{\frac{\beta(2\beta-3)}{8(1-\beta)^2}\omega^4h^3t}\right|-1+O(h^4).\end{aligned}$$

Since our interest is in the local error, we choose $\boldsymbol{t=h}$ and examine the case where x is very small, $e\hat{}x\ \approx 1+x$ then

$$\begin{aligned}|A_+| = |y_{sol}|-|u_{sol}| &= 1+\frac{\beta(2\beta-3)}{8(1-\beta)^2}\omega^4h^4-1+O(h^6)\\ &= \frac{\beta(2\beta-3)}{8(1-\beta)^2}\omega^4h^4+O(h^6)\,.\end{aligned}$$

3.2. Summary and Numerical Test

For clarity and ease of comparison, the critical characteristics of these schemes specifically their order of accuracy and leading error terms are collated in Table 1.

Table 1. Summary of phase and amplitude error of time filter used with Leapfrog scheme

Filter Scheme	Order of Accuracy	Leading Amplitude Error Term	Leading Phase Speed Error Term
LF-RA	1	$\frac{-\nu}{2(2-\nu)}(\omega h)^2$	$\frac{1+\nu}{3(2-\nu)}(\omega h)^{\,2}$
LF-RAW	1 or 2	$\frac{\nu(2\alpha-1)}{2(2-\nu)}(\omega h)^{\,2}$	$-\frac{3\nu^2(\alpha-1)(2\alpha-1)+(\nu-3\nu\alpha-2)(2-\nu)}{6(2-\nu)^2}(\omega h)^2$
LF-hoRA	2 or 3	$\frac{\beta(2\beta-3)}{8(1-\beta)^2}(\omega h)^{\,4}$	$\frac{2-5\beta}{12(1-\beta)}(\omega h)^{\,2}$

The standard method for assessing phase and amplitude errors in time-stepping schemes for non-dissipative systems is to analyze the solutions of the oscillation equation [9]

$$u'(t)=i\omega u(t) \text{ and } u(0)=1$$

where $\omega=1$ and exact solution is $u(t)=\exp{(i\omega t)}$. We use optimal parameter configurations for LF-RA ($\nu=0.1$), LF-RAW ($\nu=0.1,\ \alpha=0.53$) and LF-hoRA ($\beta=0.4$). A numerical comparison of the time filters applied to the oscillation test problem—specifically with $\Delta t=0.01$ is presented in Figures 1 and 2 and Table 2. Additionally, Figure 3 with $\Delta t=0.1$ details the amplitude evolution over the interval $t\in[0,20]$ with a magnified view of the period $t\in[15,20]$.

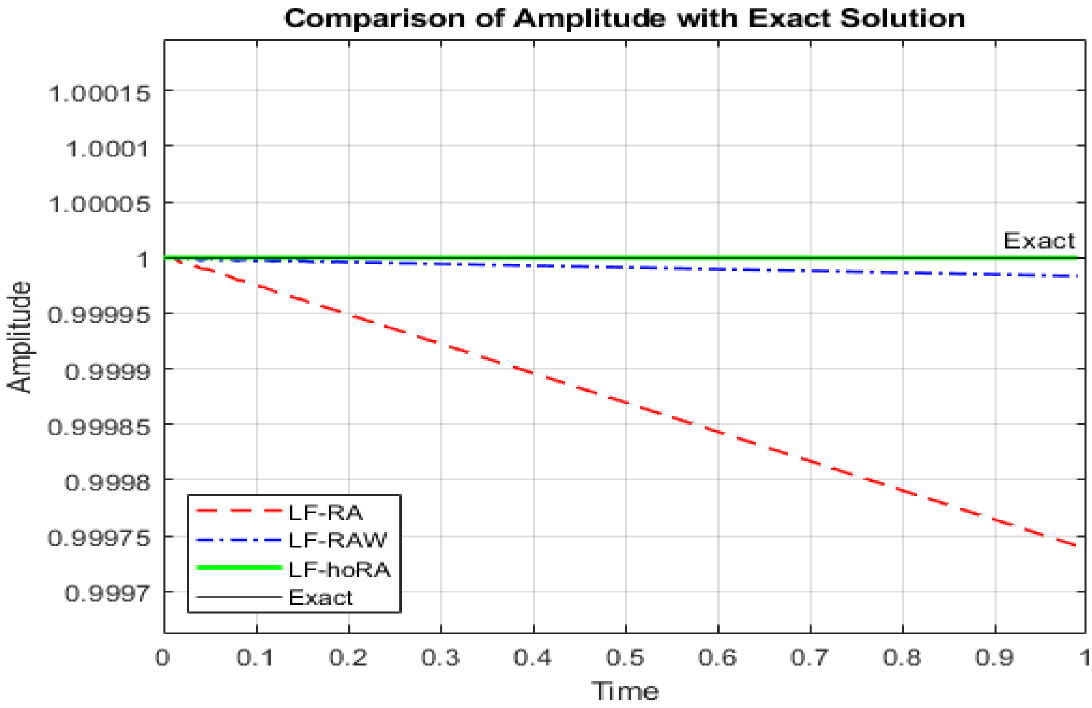

Figure 1. Comparison of amplitude of time filters compared to exact solution of oscillation equation for $\Delta t = 0.01$ *from t=0 to 1.*

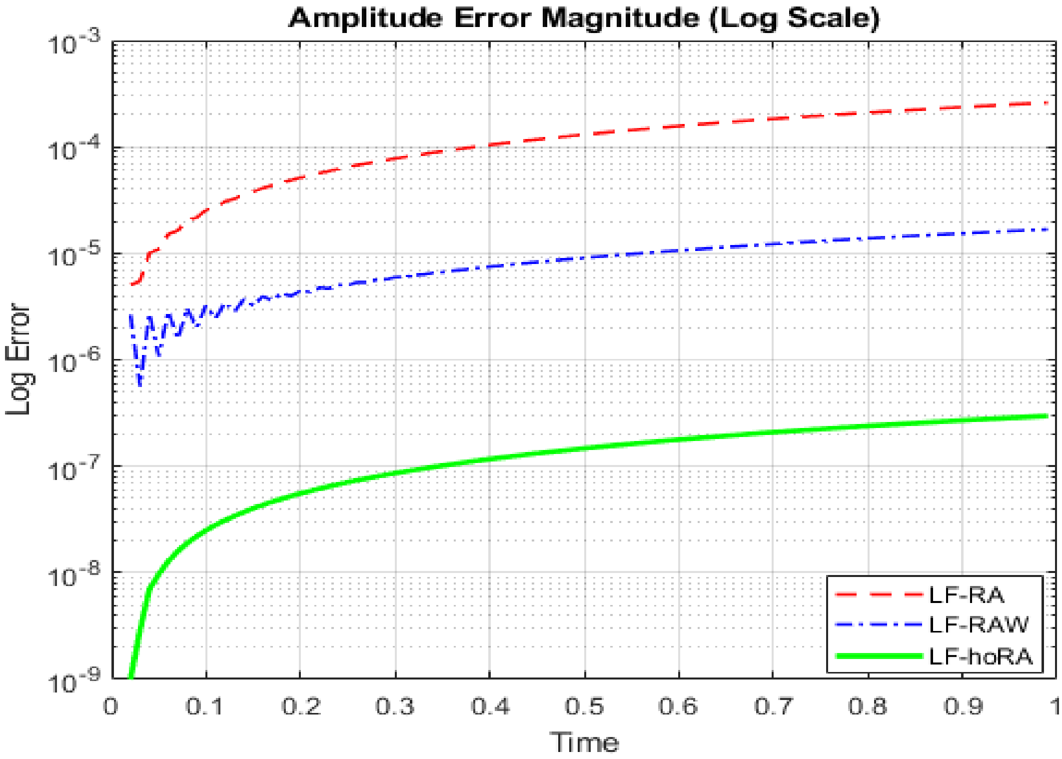

Figure 2. Comparison of amplitude error magnitude for oscillation equation for Δt=0.01 from t=0 to 1.

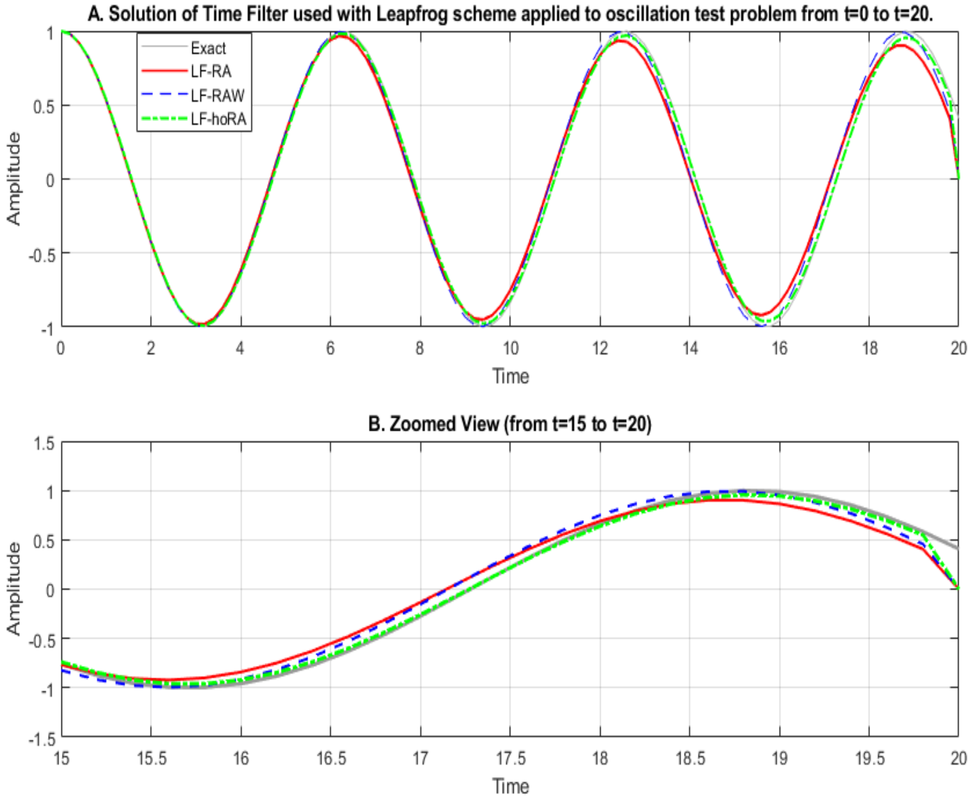

Figure 3. The exact solution of oscilattion equation with Δt=0.1 compared to time filter with ν=0.1(RA filter parameter),α=0.53(RAW filter parameter),β=0.4(hoRA filter parameter)

Table 2. Numerical error comparison of time filter applied to oscillation equation with $\Delta t = 0.01$.

Method	**Amplitude Error(** $\boldsymbol{\Delta t = 0.01}$**)**	**Phase Error(** $\boldsymbol{\Delta t = 0.01}$**)**
LF-RA	2.59115e-04	1.88079e-05
LF-RAW	1.67784e-05	1.76016e-05
LF-hoRA	2.96979e-07	1.08442e-07

4. CONCLUSION

Accurate and efficient time-stepping is crucial for reliable weather and climate simulations and remains an active research area. This work focuses on time filters applied to the widely used leapfrog scheme, analyzing their phase and amplitude errors via modified equations for the RA, RAW, and hoRA filters. By connecting filter design to modified equation analysis, this work offers practical guidance for enhancing the reliability and performance of leapfrog-based numerical schemes.

Statement of Research and Publication Ethics

The study is complied with research and publication ethics.

Artificial Intelligence (AI) Contribution Statement

This manuscript was entirely written, edited, analyzed, and prepared without the assistance of any artificial intelligence (AI) tools. All content, including text, data analysis, and figures, was solely generated by the authors.

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