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RULED INVARIANTS AND RULED SURFACES CREATED WITH SPHERICAL CURVES IN GALILEAN SPACE

Keziban ORBAY [1*], Duygu AYDOĞAN [2], Tevfik ŞAHİN [3]

[1] *Amasya University, Mathematics Education Department, Amasya, Türkiye*
[2] *Amasya University, Institute of Science, Amasya, Türkiye*
[3] *Amasya University, Mathematics Department, Amasya, Türkiye*
*** Corresponding Author:** *keziban.orbay@amasya.edu.tr*

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ABSTRACT

This study aims to examine the ruled invariants of ruled surfaces in three-dimensional Galilean space G_3, where the direction vector is determined by a curve lying on the Galilean central unit sphere. The main goal is to derive ruled invariants of such surfaces by employing a geometric approach rooted in the properties of spherical curves. To achieve this, we first compute the orthonormal frame and corresponding derivative equations of a curve lying on the surface of the Galilean central unit sphere. Then, structure functions and ruled invariants are defined and obtained in Galilean geometry sense. The study covers all three types of ruled surfaces in Galilean space. Additionally, the relationships between the Frenet frames of the curves and those of the ruled surfaces are examined in a systematic manner. The findings provide insight into the intrinsic properties of ruled surfaces and contribute to the broader understanding of geometry in non-Euclidean settings.

1. INTRODUCTION

Contemporary mathematical and mathematical physics theories have adopted a hypothetical-deductive approach, representing a departure from early theories rooted in hypotheses that were often constrained by observations or inferences, and deemed true only if they yielded correct results [1]. This departure from tradition is particularly notable in the realm of geometry. After approximately 2000 years of Euclidean geometry dominating, some non-Euclidean geometries were introduced from the 19th century onwards [2]. Einstein's theory of general relativity demonstrated that Euclidean geometry did not meet the needs of physical space [3]. Galilean geometry is related to this concept and allows physical events to be modeled geometrically. In particular, the geometric formulation of Newtonian mechanics is based on this understanding [4]. Galilean geometry's physical foundations were laid by Yaglom [5] and developed by Röschel [6]. More recent contributions further investigate geometric constructions such as parallel transport in Galilean space, particularly with respect to Frenet and Darboux frames [7]. More recently, studies on the differential geometry of curves and surfaces in Galilean and pseudo-Galilean spaces have established comprehensive frameworks that contribute to modern geometric analysis. In this regard, the works of Erjavec [8] and Dede [9] have played a crucial role in advancing the theory.

Classical differential geometry delves into the intrinsic qualities of curves and surfaces at a local level. Local properties, in this context, refer to characteristics that are determined solely by the behavior of the curve or surface in the immediate vicinity of a specific point. In contrast, global differential geometry explores how these local properties exert influence on the overall behavior of the entire curve or surface [10]. The Frenet frame, curvature, and torsion serve to characterize a curve at a specific point, whereas for a surface, the Frenet frame, distribution parameter, and curvatures at a particular point play a similar role in characterization. Surfaces come in various types, but one particularly favored for study is the ruled surface. This type of surface is generated by the continuous motion of a line based on a curve within three-dimensional space. In the context of Galilean space G_3, while various surface types such as tubular surfaces [11] and translation surfaces [12] have been extensively studied, ruled surfaces possess a unique geometry due to their generators. In the context of 3-dimensional Euclidean and Minkowski space, the structural functions of ruled surfaces are defined, and certain geometric properties are elucidated [13-15]. The deep relationship between the structure functions and the kinematical characterization of the structure functions of the non-developable ruled surface are also given [14, 16]. The structural features of ruled surfaces, termed "ruled invariants," are pivotal in the study of such surfaces. The notion of ruled invariants is deeply rooted in the kinematic interpretation of ruled surfaces. In particular, the foundational work of O. Giering established a systematic relationship between the differential invariants of ruled surfaces and the kinematics of line motions, providing a rigorous theoretical framework for their classification. Giering's approach has played a central role in the development of modern theories of ruled surface kinematics and serves as a fundamental reference for the invariant-based analysis adopted in the present study [17]. Investigating ruled surfaces and their associated ruled invariants with respect to a space curve has yielded valuable insights. The prerequisites for a ruled surface to qualify as a principal normal ruled surface of a space curve are established through the application of the theory of ruled invariants within three-dimensional Euclidean space [15]. In addition, a total classification is presented using ruled invariants theories in three-dimensional Euclidean space [18].

This study is dedicated to deriving ruled invariants for surfaces that are generated by using a curve on the Galilean unit sphere for the direction vector of generator within Galilean space G_3. Initially, we establish fundamental concepts concerning curves and ruled surfaces in Galilean space G_3. Subsequently, through the computation of Frenet frames and derivative equations for spherical curves in Galilean space G_3, we deduce several theorems and outcomes. We define the structure functions and ruled invariants of ruled surfaces in the Galilean sense, ultimately yielding ruled invariants of ruled surfaces in Galilean space, where direction vector of the generator is taken using a curve lying on the Galilean central unit sphere surface. Lastly, we provide examples of the three types of ruled surfaces to illustrate the theoretical operations.

2. PRELIMINARIES

This section encompasses fundamental definitions and concepts that hold true within Galilean space G_3. First, the basic operations will recall and then the basic information used for curves and ruled surfaces are presented.

The scalar product of two vectors $\boldsymbol{u} = (\boldsymbol{\wp_1}, \boldsymbol{\varsigma_1}, \boldsymbol{\xi_1})$ and $\boldsymbol{v} = (\boldsymbol{\wp_2}, \boldsymbol{\varsigma_2}, \boldsymbol{\xi_2})$ given in Galilean space $\boldsymbol{G_3}$ is defined by

$$\langle u, v\rangle_G = \begin{cases} \wp_1\wp_2, & \wp_1 \neq 0 \; or \; \wp_2 \neq 0 \\ \varsigma_1\varsigma_2 + \xi_1\xi_2, & \wp_1 = 0 \; and \; \wp_2 = 0 \end{cases},$$

and the cross product of these vectors too is defined by

$$u \times_G v = \begin{cases} \begin{vmatrix} 0 & e_2 & e_3 \\ \wp_1 & \varsigma_1 & \xi_1 \\ \wp_2 & \varsigma_2 & \xi_2 \end{vmatrix}, & \wp_1 \neq 0 \; or \; \wp_2 \neq 0 \\ \begin{vmatrix} e_1 & e_2 & e_3 \\ 0 & \varsigma_1 & \xi_1 \\ 0 & \varsigma_2 & \xi_2 \end{vmatrix}, & \wp_1 = 0 \; and \; \wp_2 = 0 \end{cases},$$

[5]. A vector is termed isotropic if its first component is zero; otherwise, it is referred to as non-isotropic [19].

2.1. Curve in Galilean Space G_3

In Galilean space $\boldsymbol{G_3}$, let a differentiable function

$$\alpha: I \subset R \to G_3, \alpha(t) = \big(\wp(t), \varsigma(t), \xi(t)\big),$$

be given. The set of points $\boldsymbol{\alpha}(\boldsymbol{t}) \subset \boldsymbol{G_3}$ is called a curve in Galilean space $\boldsymbol{G_3}$. Here $\boldsymbol{t} \in \boldsymbol{I}$ and the differentiable functions $\boldsymbol{\wp}, \boldsymbol{\varsigma}, \boldsymbol{\xi}: \boldsymbol{I} \to \boldsymbol{R}$ are called coordinate functions of the curve α [6]. A curve α is labeled as a regular curve when its tangent vector is $\boldsymbol{\wp}'(\mathbf{t}) \neq \mathbf{0}$, and it is identified as a unit speed curve when the norm of the tangent vector is $\|\boldsymbol{\alpha}'(\mathbf{t})\|_{\mathbf{G}} = \mathbf{1}$ [6]. The arc length of the curve α is obtained as $\boldsymbol{\wp}$ [6].

The sphere of Galilean space G_3 is defined as the set of points equidistant from a fixed point Γ_0 and it is represented by

$$\langle \Gamma - \Gamma_0, \Gamma - \Gamma_0 \rangle_G = \rho^2.$$

If the center of the Galilean sphere is the origin and its radius is 1, it is called the central unit sphere [5]. While the tangent vector of the curve drawn by a unit vector $\boldsymbol{\alpha}(\boldsymbol{s})$ on the surface of the unit sphere is $\boldsymbol{T}(\boldsymbol{s}) = \boldsymbol{\alpha}'(\boldsymbol{s})$, the orthonormal system $\{\boldsymbol{T}(\boldsymbol{s}), \boldsymbol{N}(\boldsymbol{s}), \boldsymbol{B}(\boldsymbol{s})\}$ is called the Frenet frame of the spherical curve, where $\boldsymbol{N}(\boldsymbol{s}) = \boldsymbol{\alpha}(\boldsymbol{s})$ and $B(s)=N(s)\times_{\boldsymbol{G}} \boldsymbol{T}(\boldsymbol{s})$ are [16]. Moreover, $\boldsymbol{\kappa_g}$ is the geodesic curvature, $\boldsymbol{\kappa_n}$ is the normal curvature and $\boldsymbol{\tau_\alpha}$ is the geodesic torsion at each point of the curve α in $\boldsymbol{G_3}$ which are given by

$$\kappa_g(\wp) = \langle \alpha''(\wp), B(\wp) \rangle_G,\ \kappa_n(\wp) = \langle \alpha''(\wp), N(\wp) \rangle_G, \text{and } \tau_\alpha(\wp) = \langle N'(\wp), B(\wp) \rangle_G,$$

[20].

2.2. Ruled Surface in Galilean Space G_3

If a surface can be obtained by moving a line $\boldsymbol{l}$ along a curve $\boldsymbol{r}$, this surface is called a ruled surface in Galilean space $\boldsymbol{G_3}$. Where, the line $\boldsymbol{l}$ is called generator of the ruled surface and the curve $\boldsymbol{r}$ is called the base curve of the ruled surface [6].

In Galilean space G_3, a ruled surface is parametrized by

$$\varphi(\wp, v) = r(\wp) + v\alpha(\wp);\ r, \alpha \in C^3, v \in R.$$

The base curve r and direction vector of the generator line l parameterized according to the arc length in Galilean space G_3 are given by

$$r(\wp) = (\wp, \varsigma(\wp), \xi(\wp)), \text{ and}$$
$$\alpha(\wp) = (1, \alpha_2(\wp), \alpha_3(\wp)) \text{ or } \alpha(\wp) = (0, \alpha_2(\wp), \alpha_3(\wp)), |\alpha_2^2(\wp) + \alpha_3^2(\wp)| = 1,$$

respectively [6]. There are three different types of ruled surfaces in Galilean space G_3: type A, type B and type C. The information below and more for this ruled surface can be found in Röschel's book *Die Geometrie Des Galileischen Raumes* [6].

3. MATERIAL AND METHOD

This section presents several theorems and corollaries derived from the computation of Frenet frames and derivative equations for spherical curves in Galilean space G_3. Additionally, it introduces the structure functions and ruled invariants of ruled surfaces in the Galilean sense.

3.1. The Frenet Frame of Spherical Curves in Galilean Space G_3

The central unit sphere surface in Galilean space G_3 is given by

$$S_{G_3}^2 = \{\Gamma(\wp, \varsigma, \xi) \in G_3 \,|\, \langle \Gamma, \Gamma \rangle_G = 1\}.$$

If $\wp \neq 0$, then $\langle \Gamma, \Gamma \rangle_G = \wp^2 = 1$. If $\wp = 0$, then $\langle \Gamma, \Gamma \rangle_G = \varsigma^2 + \xi^2 = 1$.

Theorem 3.1. Let $\wp^2(\wp)=1$ be the central unit sphere equation in Galilean space G_3. The Frenet vectors of a unit speed spherical curve α which is equation $\alpha(\wp)=\big(\mp 1,\varsigma(\wp),\xi(\wp)\big)$ are given by

$$T(\wp)=\big(0,\varsigma'(\wp),\xi'(\wp)\big)$$
$$N(\wp)=\big(\mp 1,\varsigma(\wp),\xi(\wp)\big)$$
$$B(\wp)=\big(0,\mp\xi'(\wp),\pm\varsigma'(\wp)\big)$$

and Frenet formulas are in matrix form as follows

$$\begin{pmatrix} T'(\wp)\\ N'(\wp)\\ B'(\wp)\end{pmatrix}=\begin{pmatrix} 0 & 0 & \mp\kappa_g(\wp)\\ 1 & 0 & 0\\ \pm\kappa_g(\wp) & 0 & 0\end{pmatrix}\begin{pmatrix} T(\wp)\\ N(\wp)\\ B(\wp)\end{pmatrix}.$$

Proof. For the Frenet vectors *T, N* and *B*, since

$$\langle T(\wp),T(\wp)\rangle_G=\varsigma'^2(\wp)+\xi'^2(\wp)=1$$
$$\langle N(\wp),N(\wp)\rangle_G=1$$
$$\langle B(\wp),B(\wp)\rangle_G=\xi'^2(\wp)+\varsigma'^2(\wp)=1$$
$$\langle T(\wp),N(\wp)\rangle_G=\langle N(\wp),B(\wp)\rangle_G=\langle B(\wp),T(\wp)\rangle_G=0,$$

frame $\{T, N, B\}$ forms an orthonormal frame. When the derivatives of the Frenet vectors of the curve are considered, we obtain

$$T'(\wp)=\big(0,\varsigma''(\wp),\xi''(\wp)\big)$$
$$N'(\wp)=\big(0,\varsigma'(\wp),\xi'(\wp)\big)$$
$$B'(\wp)=\big(0,\mp\xi''(\wp),\pm\varsigma''(\wp)\big)$$

If the derivative is taken in the equation $\langle T(\wp),T(\wp)\rangle_G=1$, we get

$$\varsigma'(\wp)\varsigma''(\wp)+\xi'(\wp)\xi''(\wp)=0. \tag{1}$$

Similarly, we have

$$\langle T'(\wp),N(\wp)\rangle_G=0$$
$$\langle T'(\wp),B(\wp)\rangle_G=\mp\xi'(\wp)\varsigma''(\wp)\pm\varsigma'(\wp)\xi''(\wp)$$
$$\langle N'(\wp),T(\wp)\rangle_G=1$$
$$\langle N'(\wp),N(\wp)\rangle_G=0$$
$$\langle N'(\wp),B(\wp)\rangle_G=0$$
$$\langle B'(\wp),T(\wp)\rangle_G=\mp\varsigma'(\wp)\xi''(\wp)\pm\xi'(\wp)\varsigma''(\wp)$$
$$\langle B'(\wp),N(\wp)\rangle_G=0$$
$$\langle B'(\wp),B(\wp)\rangle_G=0.$$

If we take the first and second derivatives of the curve $\alpha(\wp)=\big(\mp 1,\varsigma(\wp),\xi(\wp)\big)$, then we get

$$\alpha'(\wp)=\big(0,\varsigma'(\wp),\xi'(\wp)\big)=T(\wp),\text{ and }\alpha''(\wp)=\big(0,\varsigma''(\wp),\xi''(\wp)\big).$$

So, the geodesic curvature, the normal curvature and the geodesic torsion of the curve α are

$$\kappa_g(\wp) = \mp\xi'(\wp)\varsigma''(\wp) \pm \varsigma'(\wp)\xi''(\wp), \kappa_n(\wp) = 0, \tau_\alpha(\wp) = 0,$$

respectively. Thus, the proof is complete.

Theorem 3.2. Let $\wp^2(\wp) = 1$ be the central sphere equation in Galilean space G_3.The curve $\alpha(\wp) = \big(\mp 1, \varsigma(\wp), \xi(\wp)\big)$ on the sphere is a geodesic curve in the Galilean sense if and only if the functions $\varsigma'(\wp)$ and $\xi'(\wp)$ are linearly dependent.

Proof. Let the curve $\alpha(\wp) = \big(\mp 1, \varsigma(\wp), \xi(\wp)\big)$ on the sphere be a geodesic curve in the Galilean sense. α is a geodesic curve and using also (1), we have $\varsigma''(\wp) = 0$ and $\xi''(\wp) = 0$. That is

$\varsigma'(\wp) = c; c \in R$ and $\xi'(\wp) = d; d \in R$.

In this case

$$\frac{\varsigma'(\wp)}{\xi'(\wp)} = \frac{c}{d}.$$

So, $\varsigma'(\wp)$ and $\xi'(\wp)$ are linearly dependent.

Conversely, let $\varsigma'(\wp)$ and $\xi'(\wp)$ be linearly dependent. In this case, there is $\lambda \in R$ such that $\varsigma'(\wp) = \lambda\xi'(\wp)$. This means $\kappa_g(\wp) = 0$. So, the curve α becomes a geodesic curve.

Corollary 3.1. Let $\wp^2(\wp) = 1$ be the central unit sphere equation in Galilean space G_3. The curve $\alpha(\wp) = \big(\mp 1, \varsigma(\wp), \xi(\wp)\big)$ on the sphere is a curvature line in the Galilean sense.

Theorem 3.3. Let $\varsigma^2(\wp) + \xi^2(\wp) = 1$ be the central unit sphere equation in Galilean space G_3, such that $(\varsigma'(\wp)\xi(\wp) - \varsigma(\wp)\xi'(\wp))^2 = 1$ and $\varsigma'(\wp)\varsigma(\wp) + \xi'(\wp)\xi(\wp) = 0$. The Frenet vectors of a unit speed spherical curve α which is equation $\alpha(\wp) = \big(0, \varsigma(\wp), \xi(\wp)\big)$ are given by

$$T(\wp) = (0, \pm\xi(\wp), \mp\varsigma(\wp)),$$
$$N(\wp) = \big(0, \varsigma(\wp), \xi(\wp)\big),$$
$$B(\wp) = (\pm 1,0,0),$$

and Frenet formulas in matrix form are given as follows:

$$\begin{pmatrix} T'(\wp) \\ N'(\wp) \\ B'(\wp) \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} T(\wp) \\ N(\wp) \\ B(\wp) \end{pmatrix}.$$

Proof. Since $\alpha(\wp)$ is a unit speed curve, $\varsigma'^2(\wp) + \xi'^2(\wp) = 1$. Considering the inner products of Frenet vectors, we obtain

$$\langle T(\wp), T(\wp)\rangle_G = \varsigma'^2(\wp) + \xi'^2(\wp) = 1,$$

$$\langle N(\wp), N(\wp)\rangle_G = \varsigma^2(\wp) + \xi^2(\wp) = 1,$$

$$\langle B(\wp), B(\wp)\rangle_G = (\varsigma'(\wp)\xi(\wp) - \varsigma(\wp)\xi'(\wp))^2 = 1, \quad (2)$$

$$\langle T(\wp), N(\wp)\rangle_G = \varsigma'(\wp)\varsigma(\wp) + \xi'(\wp)\xi(\wp) = 0, \quad (3)$$

$$\langle N(\wp), B(\wp)\rangle_G = \langle B(\wp), T(\wp)\rangle_G = 0.$$

Thus $\{T, N, B\}$ is an orthonormal frame. According to (2), binormal vector B should be $B(\wp) = ((\pm 1,0,0)$. That is

$$\varsigma'(\wp)\xi(\wp) - \varsigma(\wp)\xi'(\wp) = \pm 1$$

Using (3), we get

$$\varsigma'(\wp) = \pm\xi(\wp) \text{ and } \xi'(\wp) = \mp\varsigma(\wp). \tag{4}$$

Accordingly, when the equations are rearranged, we obtain the Frenet vectors. So, we have the Frenet formulas

$$T'(\wp) = \big(0, \mp\varsigma(\wp), \mp\xi(\wp)\big),$$
$$N'(\wp) = \big(0, \pm\xi(\wp), \mp\varsigma(\wp)\big),$$
$$B'(\wp) = (0,0,0).$$

The geodesic curvature, the normal curvature and the geodesic torsion of the curve α are

$$\kappa_g(\wp) = 0, \kappa_n(\wp) = -1 \text{ and } \tau_\alpha(\wp) = 0,$$

respectively. Thus, the proof is complete.

Corollary 3.2. Let $\varsigma^2(\wp) + \xi^2(\wp) = 1$ be the central unit sphere equation in Galilean space G_3, such that $(\varsigma'(\wp)\xi(\wp) - \varsigma(\wp)\xi'(\wp))^2 = 1$ and $\varsigma'(\wp)\varsigma(\wp) + \xi'(\wp)\xi(\wp) = 0$. The curve $\alpha(\wp) = \big(0, \varsigma(\wp), \xi(\wp)\big)$ on the sphere is both a curvature line in the Galilean sense and a geodesic curve in the Galilean sense.

3.2. Ruled Invariants and Ruled Surfaces Created with Spherical Curves in G_3

Consider a ruled surface in the Galilean space G_3 given by $\varphi(\wp, v) = c(\wp) + v\alpha(\wp)$, where the mapping $c: I \to G_3$; $c(\wp) = \big(c_1(\wp), c_2(\wp), c_3(\wp)\big)$ serves as the base curve, and the vector field $\alpha(\wp) = \big(\wp(\wp), \varsigma(\wp), \xi(\wp)\big)$ represents the director curve of the rulings, assumed to lie on the Galilean central unit sphere $S^2_{G_3}$. If the base curve $c(p)$is taken as the striction line of the ruled surface, then it necessarily satisfies the orthogonality condition $\langle c'(\wp), \alpha'(\wp)\rangle_G = 0$. Under this assumption, the derivative of $c(p)$may be expressed with respect to the Frenet frame of the curve as

$$c'(\wp) = \lambda(\wp)N(\wp) + \mu(\wp)B(\wp),$$

where $\lambda(\wp)$ and $\mu(\wp)$ denote scalar functions defined along the base curve. This formulation motivates the following definitions in the context of Galilean differential geometry.

Definition 3.1. For the ruled surface $\varphi(\wp, v)$ in Galilean space G_3, the functions $\lambda(\wp)$, $\mu(\wp)$, and $\kappa_g(\wp)$ are referred to as the *structure functions*. The collection $\{\lambda(\wp), \mu(\wp), \kappa_g(\wp)\}$ determines the ruled surface uniquely up to Galilean transformations.

Definition 3.2. The function $\lambda(\wp)$ is designated as the *translation density* of the rulings, the function $\mu(\wp)$ is termed the *distance density*, and $\kappa_g(\wp)$ is called the *self-spinning density* associated with the ruled surface in G_3.

Definition 3.3. Differential forms $\Lambda = \lambda(\wp)d\wp$, $M = \mu(\wp)d\wp$ together with the second-order form $K = \kappa_g(\wp)d\wp^2$ are identified as the ruled invariants of the surface $\varphi(\wp, v)$ in G_3.

The ruled invariants defined in this study, in particular the self-spinning density κ_g, reflect the kinematic character of line motion in the Galilean space G_3. When a ruled surface is regarded as the trajectory of a line moving along a base curve, the invariant κ_g represents the instantaneous rate of rotation of this line about the striction line during its motion. While the curvature and torsion of the base curve determine the translational and bending behavior of the surface in space, κ_g measures the intrinsic twisting effect associated with the generator motion. An increase in κ_g indicates a more pronounced angular variation of the generator vector relative to the surface normal frame. This observation demonstrates that the obtained algebraic results are kinematically consistent with the classical theory of line geometry, in which line motion is decomposed into translational and rotational components.

4. RESULTS

This section focuses on obtaining the ruled invariants for all three types of ruled surfaces in Galilean space G_3, where direction vector of the generator is taken using a curve lying on the Galilean central unit sphere surface.

4.1. Ruled Invariants of Ruled Surface of Type A

We take the curve $\alpha: I \subset G_3 \to S^2_{G_3}$; $\alpha(\wp) = (1, \varsigma(\wp), \xi(\wp))$ on the central unit sphere surface which is equation $\wp^2(\wp) = 1$ and unit speed base curve $c: I \subset R \to G_3$; $c(\wp) = (\wp, c_2(\wp), c_3(\wp))$ in Galilean space G_3.

The ruled surface $\varphi(\wp, v) = (\wp + v, c_2(\wp) + v\varsigma(\wp), c_3(\wp) + v\xi(\wp))$ is a ruled surface of type A. After this, we will show this ruled surface with $\varphi_A(\wp, v)$.

Since $\langle c'(\wp), \alpha'(\wp)\rangle_G = 0$ for $\forall \wp \in I$, base curve $c(\wp)$ is striction line of ruled surface $\varphi_A(\wp, v)$. The Frenet vectors of the ruled surface $\varphi_A(\wp, v)$ are given by

$$t(\wp) = (1, \varsigma(\wp), \xi(\wp)),$$
$$n(\wp) = \frac{1}{\kappa(\wp)}(0, \varsigma'(\wp), \xi'(\wp)),$$
$$b(\wp) = \frac{1}{\kappa(\wp)}(0, -\xi'(\wp), \varsigma'(\wp)),$$

and Frenet formulas can be written by

$$t'(\wp) = \kappa(\wp)n(\wp),$$
$$n'(\wp) = \tau(\wp)b(\wp),$$
$$b'(\wp) = -\tau(\wp)n(\wp),$$

where the curvature $\kappa(\wp) \neq 0$ and torsion $\tau(\wp)$ of the ruled surfaces $\varphi_A(\wp, v)$ are

$$\kappa(\wp) = \sqrt{\varsigma'^2(\wp) + \xi'^2(\wp)}, \tau(\wp) = \frac{\varsigma'(\wp)\xi''(\wp) - \varsigma''(\wp)\xi'(\wp)}{\kappa^2(\wp)},$$

[6].

Theorem 4.1. Between Frenet vectors of the ruled surface $\varphi_A(\wp, v)$ and Frenet vectors of the curve $\alpha(\wp)$ the following equations hold

$$T(\wp) = n(\wp), N(\wp) = t(\wp), B(\wp) = -b(\wp),$$

$$T'(\wp) = n'(\wp), N'(\wp) = t'(\wp), -\tau(\wp)N'(\wp) = b'(\wp).$$

Theorem 4.2. The curvature of the ruled surface $\varphi_A(\wp, v)$ and the geodesic curvature of the curve $\alpha(\wp)$ are $\kappa(\wp) = 1$ and $\kappa_g(\wp) = -\tau(\wp)$, respectively.

Proof. Since the curve $\alpha(\wp) = (1, \varsigma(\wp), \xi(\wp))$ is a unit speed curve

$$\kappa(\wp) = \sqrt{\varsigma'^2(\wp) + \xi'^2(\wp)} = \|\alpha'(\wp)\|_G = 1.$$

Also, we obtain the geodesic curvature of the curve $\alpha(\wp)$ by

$$\kappa_g(\wp) = \xi'(\wp)\varsigma''(\wp) - \xi''(\wp)\varsigma'(\wp) = -\tau(\wp)\kappa^2(\wp) = -\tau(\wp).$$

Theorem 4.3. Between the tangent vector of the striction line of the ruled surface $\varphi_A(\wp, v)$ and the normal vector of direction of the ruled surface $\varphi_A(\wp, v)$ correlation is given by

$$\langle c'(\wp) - N(\wp), T'(\wp)\rangle_G = 1.$$

Proof. Since the base curve $c(\wp)$ is striction line of ruled surface $\varphi_A(\wp, v)$

$$\langle c'(\wp), \alpha'(\wp)\rangle_G = 0.$$

If we take the derivative in this equation, then

$$\langle c'(\wp), \alpha''(\wp)\rangle_G = 0, \langle c''(\wp), \alpha'(\wp)\rangle_G = 0. \tag{5}$$

From the definition of structure functions of ruled surface

$$c'(\wp) = \lambda(\wp)N(\wp) + \mu(\wp)B(\wp)$$
$$\left(1, c_2'(\wp), c_3'(\wp)\right) = \lambda(\wp)\left(1, \varsigma(\wp), \xi(\wp)\right) + \mu(\wp)(0, \xi'(\wp), -\varsigma'(\wp)).$$

We obtain

$$\lambda(\wp) = 1$$
$$c_2'(\wp) = \varsigma(\wp) + \mu(\wp)\xi'(\wp)$$
$$c_3'(\wp) = \xi(\wp) - \mu(\wp)\varsigma'(\wp)$$

and

$$c_2'(\wp)\varsigma'(\wp) - \varsigma(\wp)\varsigma'(\wp) = \xi(\wp)\xi'(\wp) - c_3'(\wp)\xi'(\wp).$$

If we take the derivative in this equation, then we have

$$c_2''(\wp)\varsigma'(\wp) + c_2'(\wp)\varsigma''(\wp) + c_3''(\wp)\xi'(\wp) + c_3'(\wp)\xi''(\wp) = \xi'^2(\wp) + \xi(\wp)\xi''(\wp) + \varsigma'^2(\wp) + \varsigma(\wp)\varsigma''(\wp),$$

$$\langle c''(\wp), \alpha'(\wp)\rangle_G + \varsigma''(\wp)\left(c_2'(\wp) - \varsigma(\wp)\right) + \xi''(\wp)\left(c_3'(\wp) - \xi(\wp)\right) = 1 \tag{6}$$

If we consider equations (5) and (6) together, the proof of the theorem is completed by

$$\varsigma''(\wp)\left(c_2'(\wp) - \varsigma(\wp)\right) + \xi''(\wp)\left(c_3'(\wp) - \xi(\wp)\right) = 1,$$
$$\langle c'(\wp) - \alpha(\wp), \alpha''(\wp)\rangle_G = 1,$$
$$\langle c'(\wp) - N(\wp), T'(\wp)\rangle_G = 1.$$

Corollary 4.1. Structure functions of the ruled surface $\varphi_A(\wp, v)$ are

$$\lambda(\wp) = 1, \mu(\wp) = \frac{c_2'(\wp) - \varsigma(\wp)}{\xi'(\wp)}, \kappa_g(\wp) = -\tau(\wp),$$

and the ruled invariants are

$$\Lambda = d\wp, M = \frac{c_2'(\wp) - \varsigma(\wp)}{\xi'(\wp)} d\wp, K = -\tau(\wp) d\wp^2.$$

4.2. Ruled Invariants of Ruled Surface of Type B

We take the curve $\alpha: I \subset G_3 \to S^2_{G_3}$; $\alpha(\wp) = \big(1, \varsigma(\wp), \xi(\wp)\big)$ on the central unit sphere surface which is equation $\wp^2(\wp) = 1$ and unit speed base curve $c: I \subset R \to G_3$; $c(\wp) = \big(0, c_2(\wp), c_3(\wp)\big)$ in Galilean space G_3.

The ruled surface $\varphi(\wp, v) = \big(v, c_2(\wp) + v\varsigma(\wp), c_3(\wp) + v\xi(\wp)\big)$ is a ruled surface of type B. After this, we will show this ruled surface with $\varphi_B(\wp, v)$.

Let the base curve $c(\wp)$ be striction line of ruled surface $\varphi_B(\wp, v)$. We have

$$(c_2{}')^2(\wp) + (c_3{}')^2(\wp) = 1, \tag{7}$$

and

$$c_2{}'(\wp)\varsigma'(\wp) + c_3{}'(\wp)\xi'(\wp) = 0. \tag{8}$$

The Frenet vectors of the ruled surface $\varphi_B(\wp, v)$ are given by

$$t(\wp) = \big(1, \varsigma(\wp), \xi(\wp)\big),$$
$$n(\wp) = \big(0, -c_3{}'(\wp), c_2{}'(\wp)\big),$$
$$b(\wp) = \big(0, c_2{}'(\wp), c_3{}'(\wp)\big),$$

and Frenet formulas can be written by

$$t'(\wp) = \kappa(\wp)n(\wp),$$
$$n'(\wp) = \tau(\wp)b(\wp),$$
$$b'(\wp) = -\tau(\wp)n(\wp),$$

where the curvature $\kappa(\wp) \neq 0$ and torsion $\tau(\wp)$ of the ruled surfaces $\varphi_B(\wp, v)$ are

$$\kappa(\wp) = -\frac{\varsigma'(\wp)}{c_3{}'(\wp)}, \tau(\wp) = \frac{c_2{}''(\wp)}{c_3{}'(\wp)},$$

[6].

Theorem 4.4. Between Frenet vectors of the ruled surface $\varphi_B(\wp, v)$ and Frenet vectors of the curve $\alpha(\wp)$ the following equations hold

$$T(\wp) = \kappa(\wp)n(\wp), N(\wp) = t(\wp), B(\wp) = \kappa(\wp)b(\wp),$$
$$T'(\wp) = \frac{\kappa_g(\wp)\kappa(\wp)}{\tau(\wp)}n'(\wp), N'(\wp) = t'(\wp), B'(\wp) = \frac{\kappa_g(\wp)\kappa(\wp)}{\tau(\wp)}b'(\wp).$$

Theorem 4.5. Structure functions of the ruled surface $\varphi_B(\wp, v)$ are

$$\lambda(\wp) = 0, \mu(\wp) = \frac{1}{\kappa(\wp)}, \kappa_g(\wp) = \tau(\wp)\kappa^2(\wp).$$

Proof. From the definition of structure functions of ruled surface

$$c'(\wp) = \lambda(\wp)N(\wp) + \mu(\wp)B(\wp),$$
$$\big(0, c_2'(\wp), c_3'(\wp)\big) = \lambda(\wp)(1, \varsigma(\wp), \xi(\wp)) + \mu(\wp)(0, \xi'(\wp), -\varsigma'(\wp)),$$

and we obtain

$$\lambda(\wp) = 0, \mu(\wp) = \frac{c_2'(\wp)}{\xi'(\wp)} = -\frac{c_3'(\wp)}{\varsigma'(\wp)} = \frac{1}{\kappa(\wp)}.$$

If we take the derivative in equation (7), then

$$c_3''(\wp) = -\frac{c_2'(\wp)c_2''(\wp)}{c_3'(\wp)} = -\tau(\wp)c_2'(\wp).$$

From equation (8)

$$\frac{c_2'(\wp)}{c_3'(\wp)} = -\frac{\xi'(\wp)}{\varsigma'(\wp)},$$

and if we take the derivative here, we complete the proof

$$\frac{\tau(\wp)c_3'(\wp)c_3'(\wp) - c_2'(\wp)\big(-\tau(\wp)c_2'(\wp)\big)}{\frac{(\varsigma')^2(\wp)}{\kappa^2(\wp)}} = \frac{\kappa_g(\wp)}{(\varsigma')^2(\wp)}$$

$$\kappa_g(\wp) = \tau(\wp)\kappa^2(\wp).$$

Corollary 4.2. The ruled invariants of the ruled surface $\varphi_B(\wp, v)$ are

$$\Lambda = 0, M = \frac{1}{\kappa(\wp)}d\wp, K = \tau(\wp)\kappa^2(\wp)d\wp^2.$$

4.3. Ruled Invariants of Ruled Surface of Type C

We take the curve $\alpha: I \subset G_3 \to S_{G_3}^2; \alpha(\wp) = \big(0, \varsigma(\wp), \xi(\wp)\big)$ on the central unit sphere surface which is equation $\varsigma^2(\wp) + \xi^2(\wp) = 1$ and unit speed base curve $c: I \subset R \to G_3$; $c(\wp) = (\wp, c_2(\wp), 0)$ in Galilean space G_3.

The ruled surface $\varphi(\wp, v) = \big(\wp, c_2(\wp) + v\varsigma(\wp), v\xi(\wp)\big)$ is a ruled surface of type C. After this, we will show this ruled surface with $\varphi_C(\wp, v)$.

Since $\langle c'(\wp), \alpha'(\wp)\rangle_G = 0$ for $\forall \wp \in I$, base curve $c(\wp)$ is striction line of ruled surface $\varphi_C(\wp, v)$. The Frenet vectors of the ruled surface $\varphi_C(\wp, v)$ are given by

$$t(\wp) = (1, c_2'(\wp), 0),$$
$$n(\wp) = \big(0, \varsigma(\wp), \xi(\wp)\big),$$
$$b(\wp) = \big(0, -\xi(\wp), \varsigma(\wp)\big),$$

and Frenet formulas can be written by

$$t'(\wp) = (0, c_2''(\wp), 0) = \kappa(\wp)(n(\wp)\cos\psi(\wp) - b(\wp)\sin\psi(\wp)),$$
$$n'(\wp) = \frac{1}{\delta(\wp)}\big(0, -\xi(\wp), \varsigma(\wp)\big),$$
$$b'(\wp) = -\frac{1}{\delta(\wp)}\big(0, \varsigma(\wp), \xi(\wp)\big),$$

where the curvature $\kappa(\wp) \neq 0$ and distribution parameter $\delta(\wp)$ of the ruled surfaces $\varphi_C(\wp, v)$ are

$$\kappa(\wp) = c_2''(\wp), \text{ and } \delta(\wp) = -\frac{\xi(\wp)}{\varsigma'(\wp)}.$$

[6]. Here, ψ is the angle between the plane $\xi = 0$ and the direction of the ruled surfaces $\varphi_C(\wp, v)$.

Theorem 4.6. Between Frenet vectors of the ruled surface $\varphi_C(\wp, v)$ and Frenet vectors of the curve $\alpha(\wp)$ the following equations hold

$$T(\wp) = -b(\wp), N(\wp) = n(\wp), N'(\wp) = -\delta(\wp)n'(\wp).$$

Theorem 4.7. Structure functions of the ruled surface $\varphi_C(\wp, v)$ are

$$\lambda(\wp) = \frac{c_2'(\wp)}{\varsigma(\wp)}, \mu(\wp) = \mp 1, \kappa_g(\wp) = 1.$$

Corollary 4.3. The ruled invariants of the ruled surface $\varphi_C(\wp, v)$ are

$$\Lambda = \frac{c_2'(\wp)}{\varsigma(\wp)} d\wp, M = \mp d\wp, K = 0.$$

Corollary 4.4. Distribution parameter of the ruled surfaces $\varphi_C(\wp, v)$ is $\delta(\wp) = \mp 1$.

The fact that the generator vector curve lies on the circular part of the Galilean unit sphere ($\wp = 0$) precludes the existence of ruled surfaces of type A and type B as a consequence of the metric structure of the Galilean space G_3. Ruled surfaces of type A and type B are characterized by non-isotropic generators possessing a non-vanishing first component aligned with the direction of absolute time; however, the condition $\wp = 0$ restricts the generator to isotropic planes, thereby eliminating the temporal component. Consequently, the rulings of the surface are confined to the Euclidean-like subspace of G_3, which leads to the formation of ruled surfaces of type C. Hence, the condition $\wp = 0$ necessitates the emergence of ruled surfaces of type C as an intrinsic requirement of the causal geometry of space.

For each of the type A, type B, and type C ruled surfaces investigated in this study, the associated surface and curve roof pairings reveal a coherent and consistent geometric structure within the Galilean space G_3. The frame correspondences between curves and surfaces in the Galilean space G_3 are strictly governed by the causal structure of the space. For ruled surfaces of type A and type B, the fundamental vectors, such as t and N, are non-isotropic vectors representing the direction of absolute time. In contrast, for ruled surfaces of type C, the vector N lies within isotropic planes characterized by a vanishing first component. This distinction is fundamental: while ruled surfaces of type A and type B surfaces involve the full Galilean metric, ruled surfaces of type C primarily exhibit geometric properties derived from the isotropic metric of the corresponding subspace. This systematic classification confirms that the ruled invariants associated with each surface type are defined in accordance with their respective isotropic or non-isotropic nature.

4.4. Examples for Ruled Surface of Type A, Type B and Type C

To visualize the theoretical approach, examples of the three types of ruled surfaces in Galilean space G_3 are provided, and the figures are drawn using Python programming language.

Example 4.1. We take the curve $\alpha: I \subset G_3 \to S^2_{G_3}$; $\alpha(\wp) = (1, 0.2sin(0.8\wp), 0.15cos(0.8\wp))$ on the central unit sphere surface, which has the equation $\wp^2(\wp) = 1$, and the unit speed base curve $c: I \subset R \to G_3$; $c(\wp) = (\wp, 0.3sin(\wp), 1.5cos(0.8\wp))$ in Galilean space G_3. If direction vector of the generator is taken as $\alpha(\wp) = (1, 0.2sin(0.8\wp), 0.15cos(0.8\wp))$, the ruled surface becomes

$$\varphi_A(\wp, v) = (\wp + v, 0.3sin(\wp) + 0.2vsin(0.8\wp), 1.5cos(0.8\wp) + 0.15vcos(0.8\wp)).$$

The base curve $c(\wp)$ is shown in red on the ruled surface $\varphi_A(\wp, v)$ in Figure 1. For better visibility of the surface, the ranges $\wp \in [0, 10]$ and $v \in [-1,1]$ were selected.

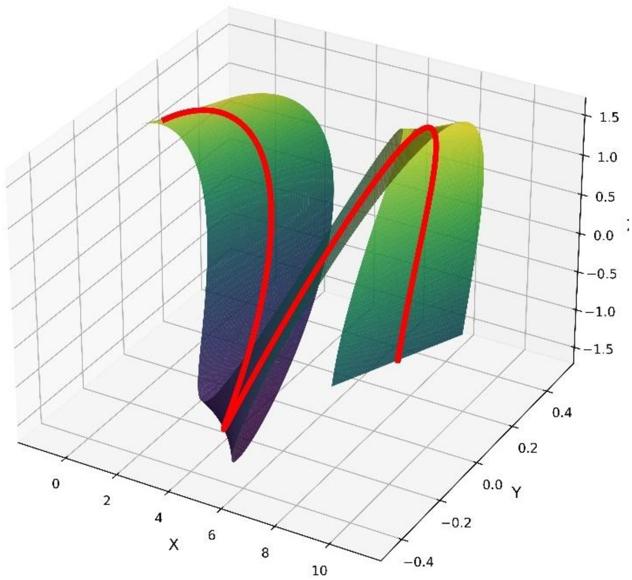

Figure 1. The ruled surface $\varphi_A(p,v)$

Example 4.2. We take the curve $\alpha: I \subset G_3 \to S^2_{G_3}$; $\alpha(p) = (1, 0.3p, 0.2p^2)$ on the central unit sphere surface, which has equation $p^2(p) = 1$ and the unit speed base curve $c: I \subset R \to G_3$; $c(p) = \left(0, \frac{\sqrt{2}}{2} sinp + \frac{\sqrt{2}}{2} cosp, \frac{\sqrt{2}}{2} cosp - \frac{\sqrt{2}}{2} sinp\right)$ in Galilean space G_3. If direction vector of the generator is taken as $\alpha(p) = (1, 0.3p, 0.2p^2)$, the ruled surface becomes

$$\varphi_B(p,v) = \left(v, \left(\frac{\sqrt{2}}{2} sinp + \frac{\sqrt{2}}{2} cosp\right) + 0.3vp, \left(\frac{\sqrt{2}}{2} cosp - \frac{\sqrt{2}}{2} sinp\right) + 0.2vp^2\right).$$

The base curve $c(p)$ is shown in red on the ruled surface $\varphi_B(p,v)$ in Figure 2. For better visibility of the surface, the ranges $p \in [-4, 4]$ and $v \in [-2,2]$ were selected.

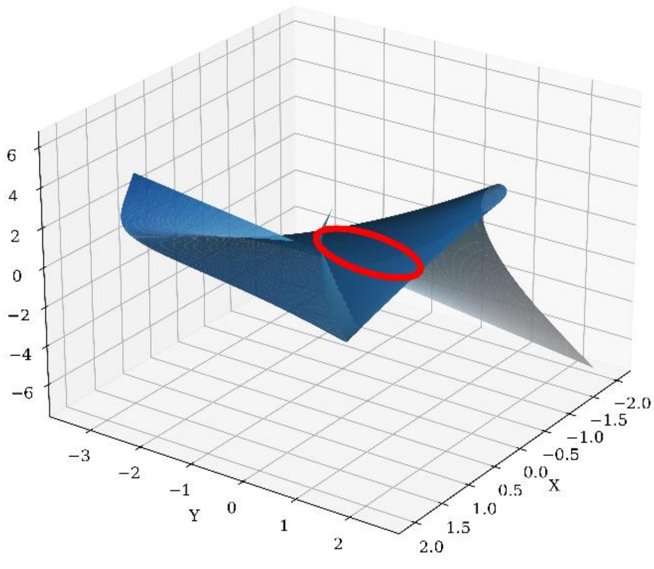

Figure 2. The ruled surface $\varphi_B(p,v)$

Example 4.3. We take the curve $\alpha: I \subset G_3 \to S^2_{G_3}$; $\alpha(p) = (0, cosp, sinp)$ on the central unit sphere surface , which has equation $\varsigma^2(p) + \xi^2(p) = 1$, and the unit speed base curve $c: I \subset R \to G_3$; $c(p) = (p, 0, 0)$ in Galilean space G_3. If direction vector of the generator is taken as $\alpha(p) = (0, cosp, sinp)$, the ruled surface becomes

$$\varphi_C(\wp, v) = (\wp, v\cos\wp, v\sin\wp).$$

The base curve $c(\wp)$ is shown in red on the ruled surface $\varphi_C(\wp, v)$ in Figure 3. For better visibility of the surface, the ranges $\wp \in [-4, 4]$ and $v \in [-2,2]$ were selected.

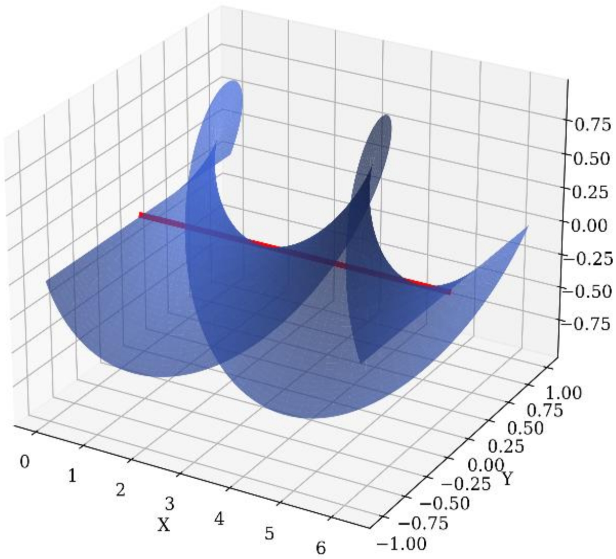

Figure 3. The ruled surface $\varphi_C(\wp, v)$

5. CONCLUSION

Ruled surfaces in Galilean space can be categorized into three distinct types based on the literature of their underlying base curves and direction vector of generators: Type A, type B, and type C. In this study, the ruled invariants of all three types of ruled surfaces are obtained.

We observe that if the curve $\alpha(\wp) = (1, \varsigma(\wp), \xi(\wp))$ is on the central unit sphere surface given by the equation $\wp^2(\wp) = 1$ and if $c(\wp) = (\wp, c_2(\wp), c_3(\wp))$ is unit speed base curve, a ruled surface of type A and its ruled invariants are obtained. Moreover, we observe that if the curve $\alpha(\wp) = (1, \varsigma(\wp), \xi(\wp))$ is on the central unit sphere surface given by the equation $\wp^2(\wp) = 1$ and if $c(\wp) = (0, c_2(\wp), c_3(\wp))$ is unit speed base curve, a ruled surface of type B and along with its respective ruled invariants are obtained. However, a ruled surface of type C is not obtained. Additionally, we observe that if the curve $\alpha(\wp) = (0, \varsigma(\wp), \xi(\wp))$ is on the central unit sphere surface given by the equation $\varsigma^2(\wp) + \xi^2(\wp) = 1$ and if $c(\wp) = (\wp, c_2(\wp), 0)$ is unit speed base curve, only a ruled surface of type C and its ruled invariants are obtained.

Conflict of Interest Statement

There is no conflict of interest between the authors.

Statement of Research and Publication Ethics

The study is complied with research and publication ethics.

Artificial Intelligence (AI) Contribution Statement

The figures are drawn using Python programming language.

Contributions of the Authors

Conceptualization, K.O. and T.Ş.; methodology, K.O. and T.Ş.; formal analysis, K.O. and D.A.; investigation, K.O. and D.A.; resources, K.O. and D.A.; data curation, K.O. and D.A. writing—original draft preparation, K.O.; writing—review and editing, K.O. All authors have read and agreed to the published version of the manuscript.

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