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MAXIMUM LIKELIHOOD ESTIMATION OF THE UNIT GOMPERTZ DISTRIBUTION USING MEDIAN RANKED SET SAMPLING

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ABSTRACT

Sampling methods are fundamental approaches that enhance the efficiency of scientific studies. However, to minimize ranking errors and obtain more accurate estimators, it is essential to develop alternative techniques to classical methods. The Median Ranked Set Sampling (MRSS) method stands out as a robust tool that minimizes ranking errors and enables more efficient evaluation of data. This method is particularly effective in improving the accuracy of sampling processes. On the other hand, the Unit-Gompertz (UG) distribution, with its flexible structure and parameters confined to the [0,1] interval, has emerged as a significant modeling option in fields such as health sciences, reliability theory, and actuarial studies. This study aims to analyze the performance of the MRSS method for the unknown parameters of the UG distribution and compare it with the Simple Random Sampling (SRS) method to develop more effective estimations. In addition to simulation results, a real-data application is also provided to demonstrate the practical usefulness of the proposed approach. The results demonstrated that MRSS provides more accurate and efficient estimates compared to SRS.

1. INTRODUCTION

The use of sampling methods instead of entire datasets provides significant advantages in research in terms of both time and cost, making them a fundamental approach for inreasing the efficiency and productivity of scientific studies. One commonly used technique within this framework is the Simple Random Sampling (SRS) method. However, to minimize ranking errors, an alternative method known as Ranked Set Sampling (RSS) was proposed by McIntyre [1]. Over the years, the method has been further developed, and various modifications have been introduced to adapt it to different measured conditions. One such modification is the Median Ranked Set Sampling (MRSS) technique, proposed by Muttlak [2], which provides more efficient and often unbiased estimators. Previous studies employing the RSS method as an alternative to the SRS method have demonstrated the superiority of RSS in producing more accurate estimators. For instance Takahasi et al. [3], Dell and Clutter [4], and Wolfe [5] showed that the RSS method outperformed SRS in terms of estimation accuracy. Hussian [6] explored the estimation of unknown parameters of the Kumaraswamy distribution using both SRS and RSS techniques, and compared the performances of maximum likelihood (ML) and Bayesian estimators in terms of bias and mean squared error.

Al-Omari [7] and Al-Omari et al. [8] utilized the RSS method to obtain entropy estimators for continuous random variables. Obeidat et al. [9] compared the performances of maximum likelihood and Bayesian estimation methods for the Gompertz distribution under both SRS and RSS frameworks. Cavdar [10] examined the unknown parameters of the Chen and Extended Logarithmic distributions using SRS, RSS, and Record Ranked Set Sampling (RRSS) methods, and evaluated the ML and Bayesian estimators through Monte Carlo simulations and relative efficiency analysis. Gul [11] considered the estimation of the parameters of the Lomax distribution using SRS, RSS, MRSS, and ERSS methods, focusing on the performance of maximum likelihood estimators.

The Gompertz distribution is a probability distribution commonly used to model mortality rates and life expectancy. It was first introduced by the English actuary and mathematician Benjamin Gompertz in 1825 [12]. This distribution assumes that the risk of death increases exponentially with age. The Unit-Gompertz (UG) distribution is a scaled version of the Gompertz distribution, where the values of x are constrained to the interval [0,1]. It is particularly suitable for situations where variables must lie within the unit interval, making it useful in survival analysis and reliability modeling studies. Several authors have investigated the UG distribution in the literature. Mazucheli et al. [13] proposed the UG distribution as an alternative to the Beta and Kumaraswamy distributions, both of which have two shape parameters. Jha et al. [14] further explored multi-component stress–strength reliability under the UG distribution with a common scale parameter using progressive Type II censoring and presented their results using three real datasets under both frequentist and Bayesian frameworks. Bantan et al. [15] introduced the unit gamma/Gompertz distribution, which is based on the inverse exponential and gamma/Gompertz families, extending flexibility to the unit interval; they performed ML estimation and assessed model performance using real datasets. Arshad et al. [16] analyzed the estimation problem of the UG distribution parameters (α and β) under a Bayesian framework using Markov Chain Monte Carlo and the Lindley approximation within the generalized order statistics structure. Alsadat et al. [17] predicted stress–strength reliability based on UG distributions using seven different estimation techniques and showed that RSS-based predictions were superior to SRS. Akata et al. [18] introduced the Kumaraswamy Unit-Gompertz distribution, derived its mathematical properties, and examined the performance of ML estimators through simulations and real data applications. Sindhu et al. [19] investigated additional statistical properties of the UG distribution and demonstrated that its flexible hazard rate shapes provide superior fit to real datasets.

In summary, the UG distribution is particularly useful in fields such as medicine and health, reliability theory, and actuarial science, and it has a scale parameter $\alpha > 0$ and a shape parameter $\beta > 0$. Its probability density function (PDF) and cumulative distribution function (CDF) are given by,

$$f(x;\alpha,\beta) = \alpha\,\beta x^{-(\beta+1)} \exp\left[-\alpha(x^{-\beta}-1)\right] \qquad \alpha,\beta,x > 0, \tag{1}$$

$$F(x|\alpha,\beta) = 1 - \exp\left[-\alpha(x^{-\beta}-1)\right] \qquad \alpha,\beta,x > 0, \tag{2}$$

MRSS is a method that provides more efficient estimations, particularly for symmetric distributions with a single mode. It is based on measuring the median values within each set. The MRSS method, proposed by Muttlak [2], involves the following step-by-step procedure for sample selection:

- Step 1: m sets, each consisting of m units, are randomly selected.
- Step 2: The units within each set are ordered according to the variable of interest.
- Step 3: The MRSS method differs based on whether the set size m is even or odd.
- If the set size m is even, the $(m/2)^{th}$smallest unit is selected from the first $(m/2)$ sets, and the $((m+1)/2)^{th}$smallest unit is selected from the next $(m/2)$sets.
- If the set size m is odd, the $((m+1)/2)^{th}$smallest unit is selected from each set.
- Step 4: This selection process is repeated r times, ensuring that the total sample size is $n = m * r$.

Illustrative Example for the MRSS Procedure:

To further clarify the selection process described above, consider a hypothetical example where the set size is $m = 4$ (an even number) and the number of cycles is $r = 2$. The target total sample size is calculated as $n = m \ x \ r = 8$. The MRSS procedure proceeds as follows:

Cycle 1: Four sets, each containing four units, are drawn independently.

Since $m = 4$ is even, the sets are divided into two groups of size $m/2 = 2$

- From the first two sets, the $(m/2)^{th}$ = 2^{nd}order statistic is selected.
- From the remaining two sets, the $((m+2)/2)^{th} = 3^{rd}$ order statistic is selected.

Cycle 2: The same procedure is repeated independently with four newly drawn sets, again selecting the 2nd and 3rd ordered units according to the MRSS rule. By the end of the two cycles, a total of $n = 8$ observations are collected.

Unlike SRS, these observations consist of median or near-median units, which substantially reduces ranking errors and improves estimation efficiency.

Several studies have also explored the MRSS method, which, although rooted in SRS, has been shown to be more effective. For instance, Omar and Ibrahim [20] compared various estimators for the shape and scale parameters of the Pareto distribution using SRS, RSS, MRSS, and ERSS methods. Koyuncu [21] demonstrates that the proposed regression estimators using the MRSS and neoteric ranked set sampling (NRSS) methods offer higher accuracy and greater efficiency compared to the classical RSS method, as shown through a case study of a rare endemic plant species from Türkiye. Hassan et al. [22] demonstrated that the MRSS method is more efficient than SRS and RSS. Gursoy et al. [23] investigated shrinkage estimators for the location parameter of the normal distribution using RSS and MRSS methods, conducting Monte Carlo simulations to compare their efficiencies under various conditions. Their findings highlighted the greater effectiveness of the MRSS method.

Studies in the literature predominantly focus on the RSS method, while the use of the UG distribution has been rarely addressed in terms of distributional analyses. The lack of research combining both the method and the distribution forms the primary motivation for this study. In this context, the MRSS method was applied to estimate the parameters of the UG distribution, and its performance was compared with the SRS method. In summary, the aim of this study is to demonstrate the potential of the MRSS method in minimizing ranking errors and its accuracy in parameter estimation for the UG distribution. Although the RSS and MRSS methods have been successfully applied to various probability distributions in the literature, to the best of our knowledge, no study has specifically investigated the estimation of the Unit-Gompertz (UG) distribution parameters using the Median Ranked Set Sampling (MRSS) method. Existing studies have generally focused on either the classical Gompertz distribution or standard RSS designs. This study aims to fill this gap by deriving the Maximum Likelihood Estimators (MLEs) for the UG distribution under the MRSS framework. Furthermore, it provides a comprehensive comparison with the Simple Random Sampling (SRS) method through Monte Carlo simulations to demonstrate the efficiency gains, particularly in minimizing ranking errors and improving estimation accuracy for bounded unit-interval data.

The study is organized into six sections, including the introduction. The second section, following the introduction, discusses the MRSS method, one of the sampling techniques used in the study. The third section addresses the UG distribution analyzed in the study, while the fourth section presents the maximum likelihood estimation and inference based on the relevant methods for the distribution. The findings from the simulation studies are presented in the fifth section, and the conclusion of the study is provided in the final section.

2. MATERIAL AND METHOD

In this section, we will examine the maximum likelihood estimates of the unknown parameters of the UG distribution under SRS and MRSS.

2.1. Notation and Symbol Definitions

For clarity and consistency throughout the manuscript, all mathematical symbols and notation used in the formulation of the UG distribution and the sampling designs are defined as follows:

α: Shape parameter of the UG distribution.

β: Scale parameter of the UG distribution.

$f(x;\ \alpha, \beta)$: Probability density function (PDF) of the UG distribution.

$F(x;\ \alpha, \beta)$: Cumulative distribution function (CDF).

$l(\alpha, \beta)$: Log–likelihood function.

n: Final sample size.

m: Set size in MRSS.

r: Number of cycles in MRSS (so that $n = mxr$).

N: Population size.

K_1, K_2: Normalizing constants appearing in the MRSS likelihood expressions.

x_k: The k-th order statistic within a set.

2.2. Estimation of Parameters Using SRS

UG distribution with parameters α and β. The likelihood function of α and β is given by

$$L(\alpha, \beta;\ x) = \prod_{i=1}^{n} \alpha\, \beta x^{-(\beta+1)} \exp\left[-\alpha(x^{-\beta} - 1)\right] \tag{3}$$

and the log-likelihood function is

$$lnL(\alpha, \beta) = nln(\alpha) + nln(\beta) - (\beta + 1) \sum_{i=1}^{n} ln\,(x_i) - \alpha \sum_{i=1}^{n} {x_i}^{-\beta} + n\alpha. \tag{4}$$

The maximum likelihood estimates of parameters are obtained by simultaneously solving the following equations:

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} {x_i}^{-\beta} + n = 0, \tag{5}$$

$$\frac{\partial l}{\partial \beta} = \frac{n}{\beta} - \sum_{i=1}^{n} \ln(x_i) + \alpha \sum_{i=1}^{n} {x_i}^{-\beta}\, ln(x_i) = 0. \tag{6}$$

2.3. Estimation of Parameters Using MRSS

(a) Consider the sample size m is odd. The probability density function of $X_{ij((m+1)/2)}$ is given by

$$g\left(x_{ij\left(\frac{m+1}{2}\right)};\ \alpha, \beta\right) = \frac{m!}{\left(\frac{m-1}{2}\right)!^2} f\left(x_{ij\left(\frac{m+1}{2}\right)};\ \alpha, \beta\right) \left(F\left(x_{ij\left(\frac{m+1}{2}\right)};\ \alpha, \beta\right)\right)^{\left(\frac{m-1}{2}\right)}$$

$$X\left(1 - F\left(x_{ij\left(\frac{m+1}{2}\right)};\ \alpha, \beta\right)\right)^{\left(\frac{m-1}{2}\right)}, \tag{7}$$

$$g\left(x_{ij\left(\frac{m+1}{2}\right)};\ \alpha, \beta\right) = \frac{m!}{\left(\frac{m-1}{2}\right)!^2} \left(\alpha\, \beta {x_{ij\left(\frac{m+1}{2}\right)}}^{-(\beta+1)} exp\left[-\alpha\left({x_{ij\left(\frac{m+1}{2}\right)}}^{-\beta} - 1\right)\right]\right)$$

$$\left(exp\left[-\alpha\left({x_{ij\left(\frac{m+1}{2}\right)}}^{-\beta} - 1\right)\right]\right)^{\left(\frac{m-1}{2}\right)} \left(1 - \left(exp\left[-\alpha\left({x_{ij\left(\frac{m+1}{2}\right)}}^{-\beta} - 1\right)\right]\right)\right)^{\left(\frac{m-1}{2}\right)}. \tag{8}$$

The likelihood function of $MRSS_{odd}$ is given by

$$L(\alpha, \beta;\ x) = \prod_{i=1}^{m} \prod_{j=1}^{r} g\left(x_{ij\left(\frac{m+1}{2}\right)};\ \alpha, \beta\right) \tag{9}$$

$$= (K_1)^{mr}\alpha^{mr}\beta^{mr}\prod_{i=1}^{m}\prod_{j=1}^{r} x_{ij\left(\frac{m+1}{2}\right)}{}^{-(\beta+1)}\, exp\left[-\alpha\left(x_{ij\left(\frac{m+1}{2}\right)}{}^{-\beta}-1\right)\right]$$

$$\times\ \left(exp\left[-\alpha\left(x_{ij\left(\frac{m+1}{2}\right)}{}^{-\beta}-1\right)\right]\right)^{\left(\frac{m-1}{2}\right)}\times\left(1-\left(exp\left[-\alpha\left(x_{ij\left(\frac{m+1}{2}\right)}{}^{-\beta}-1\right)\right]\right)\right)^{\left(\frac{m-1}{2}\right)}, \tag{10}$$

where $K_1 = \frac{m!}{\left(\frac{m-1}{2}\right)!^2}$.

The full log-likelihood expression for the $MRSS_{odd}$ case, together with its first- and second-order derivatives, involves lengthy algebraic forms. For this reason, only the essential components are presented here to maintain the readability of the main text. The complete derivations are provided in Appendix A. The maximum likelihood estimators of α and β are obtained by setting the first derivatives equal to zero; however, due to the absence of closed-form solutions, numerical optimization methods are required to solve the resulting nonlinear equations.

(b) Consider the sample size m is even. The probability density function of $X_{ij\left(\frac{m}{2}\right)}$ and $X_{ij\left(\frac{m+2}{2}\right)}$ is given by

$$g\left(x_{ij\left(\frac{m}{2}\right)}, x_{ij\left(\frac{m+2}{2}\right)}\,;\,\alpha,\beta\right) = \left(\frac{m!}{\left(\frac{m}{2}\right)!\left(\frac{m}{2}-1\right)!}\right)^2 f\left(x_{ij\left(\frac{m}{2}\right)};\,\alpha,\beta\right)\left(F\left(x_{ij\left(\frac{m}{2}\right)};\,\alpha,\beta\right)\right)^{\frac{m}{2}-1}$$

$$\times\left(1-F\left(x_{ij\left(\frac{m}{2}\right)};\,\alpha,\beta\right)\right)^{\frac{m}{2}} f\left(x_{ij\left(\frac{m+2}{2}\right)};\,\alpha,\beta\right)\left(F\left(x_{ij\left(\frac{m+2}{2}\right)};\,\alpha,\beta\right)\right)^{\frac{m}{2}}$$

$$\times\left(1-F\left(x_{ij\left(\frac{m+2}{2}\right)};\,\alpha,\beta\right)\right)^{\frac{m}{2}-1}, \tag{11}$$

$$g\left(x_{ij\left(\frac{m}{2}\right)}, x_{ij\left(\frac{m+2}{2}\right)}\,;\,\alpha,\beta\right) = \left(\frac{m!}{\left(\frac{m}{2}\right)!\left(\frac{m}{2}-1\right)!}\right)^2\left(\alpha\,\beta x_{ij\left(\frac{m}{2}\right)}{}^{-(\beta+1)} exp\left[-\alpha\left(x_{ij\left(\frac{m}{2}\right)}{}^{-\beta}-1\right)\right]\right)$$

$$\times\left(\ exp\left[-\alpha(x_{ij\left(\frac{m}{2}\right)}{}^{-\beta}-1)\right]\right)^{\left(\frac{m}{2}-1\right)}\left(1-\left(exp\left[-\alpha\left(x_{ij\left(\frac{m}{2}\right)}{}^{-\beta}-1\right)\right]\right)\right)^{\left(\frac{m}{2}\right)}$$

$$\left(\alpha\beta x_{ij\left(\frac{m+2}{2}\right)}{}^{-(\beta+1)} exp\left[-\alpha\left(x_{ij\left(\frac{m+2}{2}\right)}{}^{-\beta}-1\right)\right]\right)\left(exp\left[-\alpha\left(x_{ij\left(\frac{m+2}{2}\right)}{}^{-\beta}-1\right)\right]\right)^{\left(\frac{m}{2}\right)}$$

$$\left(1-\left(exp\left[-\alpha\left(x_{ij\left(\frac{m+2}{2}\right)}{}^{-\beta}-1\right)\right]\right)\right)^{\left(\frac{m}{2}-1\right)}. \tag{12}$$

Then, the likelihood function of $MRSS_{even}$ is

$$L(\alpha,\beta;\,x) = \prod_{i=1}^{\frac{m}{2}}\prod_{j=1}^{r} g\left(x_{ij\left(\frac{m}{2}\right)}\,;\,\alpha,\beta\right)\prod_{i=\frac{(m+2)}{2}}^{m}\prod_{j=1}^{r} g\left(x_{ij\left(\frac{m+2}{2}\right)}\,;\,\alpha,\beta\right) \tag{13}$$

$$= (K_2)^{mr}\alpha^{mr}\beta^{mr}\prod_{i=1}^{\frac{m}{2}}\prod_{j=1}^{r} x_{ij\left(\frac{m}{2}\right)}^{-(\beta+1)} exp\left[-\alpha\left(x_{ij\left(\frac{m}{2}\right)}^{-\beta}-1\right)\right]$$

$$\times \left(exp\left[-\alpha\left(x_{ij\left(\frac{m}{2}\right)}^{-\beta}-1\right)\right]\right)^{\left(\frac{m}{2}-1\right)}\left(1-\left(exp\left[-\alpha\left(x_{ij\left(\frac{m}{2}\right)}^{-\beta}-1\right)\right]\right)\right)^{\left(\frac{m}{2}\right)}$$

$$\prod_{i=\frac{(m+2)}{2}}^{m}\prod_{j=1}^{r} x_{ij\left(\frac{m+2}{2}\right)}^{-(\beta+1)} exp\left[-\alpha\left(x_{ij\left(\frac{m+2}{2}\right)}^{-\beta}-1\right)\right]$$

$$\times \left(exp\left[-\alpha\left(x_{ij\left(\frac{m+2}{2}\right)}^{-\beta}-1\right)\right]\right)^{\left(\frac{m}{2}\right)}\left(1-\left(exp\left[-\alpha\left(x_{ij\left(\frac{m+2}{2}\right)}^{-\beta}-1\right)\right]\right)\right)^{\left(\frac{m}{2}-1\right)}, \quad (14)$$

where $K_2 = \left(\frac{m!}{\left(\frac{m}{2}\right)!\left(\frac{m}{2}-1\right)!}\right)^2$.

As in the $MRSS_{odd}$ formulation for odd set sizes, the explicit log-likelihood function and the first- and second-order derivatives for the even-set case are also provided in Appendix A. These formulations are not repeated here to preserve the readability of the main text. As in the odd-case derivation, the maximum likelihood estimators are obtained by setting the score equations equal to zero; however, due to the absence of closed-form solutions, the parameter estimates for the even-case MRSS design are likewise computed using numerical optimization methods.

The Monte Carlo simulation study was carried out in the MATLAB programming environment to evaluate the performance of the MLEs. To obtain the parameter estimates for $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, the negative log-likelihood function given in Eq. (4) was minimized using the Nelder–Mead simplex algorithm implemented in the “fminsearch” function. This derivative-free optimization method was selected due to its robustness in handling nonlinear and potentially non-convex likelihood surfaces, which are common in bounded asymmetric distributions such as the Unit-Gompertz model. To reduce the likelihood of convergence to local minima and to improve numerical stability, each optimization run was initialized at the true parameter values used for data generation. A maximum limit of 10,000 function evaluations was allowed per iteration.

Across the $\boldsymbol{N = 10,000}$ replications for all sample sizes ($\boldsymbol{n = 48, 60, 72, 96}$), the algorithm converged consistently within the specified parameter space. Although the likelihood function of the UG distribution may theoretically admit multiple stationary points, no non-convergence or unstable estimation behavior was observed in the implemented simulation framework.

3. RESULTS AND DISCUSSION

In this study, the performance of ML estimators for the UG distribution was analyzed by comparing the SRS and MRSS methods. The purpose of the study was to assess the impact of these methods on estimation accuracy. The Monte Carlo simulation was conducted using MATLAB (R2021a) software to evaluate the performance of ML estimators. The simulation design involved the following specifications:

Sample Sizes: $n = 48,60,72,96$

Parameter Combinations: $\alpha = (1.0, 1.5, 2.0) and\ \beta = (0.5, 1.0, 1.5, 2.0)$

Iterations: N =10.000

To evaluate estimator performance, the bias and mean squared error (MSE) were calculated. The bias of an estimator $\hat{\theta}$ is given by

$$Bias\,(\hat{\theta}) = E(\hat{\theta}) - \theta,$$

and the mean squared error is

$$MSE\,(\hat{\theta}) \;= E[(\hat{\theta} - \theta)^2] = Var\,(\hat{\theta}) + (Bias\,(\hat{\theta})\,)^2,$$

The comparison was based on bias and mean square error criteria's. The results are given in Tables 1-3.

Table 1. Estimated bias, root mean-squared and coverage probability for the UG distribution ($\alpha = 1.0$)

			α = 1							
			MRSS				**SRS**			
β	**n**	**m;r**	**Bias**		**MSE**		**Bias**		**MSE**	
			$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
		3;16	0,1090	**0,0554**	**0,9183**	0,0382				
	48	4;12	**0,0523**	**0,0545**	**0,4571**	**0,0356**	0,1045	0,0563	1,6948	0,0367
		6;8	**0,0457**	0,0573	**0,4451**	0,0371				
		3;20	**0,0432**	0,0510	**0,4317**	0,0294				
	60	4;15	**0,0387**	0,0439	**0,3239**	**0,0276**	0,0911	0,0423	2,9607	0,0277
		6;10	**0,0409**	0,0444	**0,5094**	**0,0275**				
0,5		3;24	**0,0550**	0,0372	**0,3405**	0,0233				
	72	4;18	**0,0244**	0,0367	**0,2290**	0,0214	0,0615	0,0356	0,5613	0,0230
		6;12	**0,0321**	0,0358	**0,2476**	0,0219				
		3;32	**0,0387**	0,0273	**0,2329**	0,0168				
	96	4;24	**0,0232**	**0,0257**	**0,1611**	**0,0158**	0,0434	0,0264	0,2430	0,0166
		6;16	**0,0122**	**0,0263**	**0,1551**	**0,0160**				
		3;16	0,1193	**0,1117**	2,2674	**0,1493**				
	48	4;12	**0,0516**	**0,1133**	**0,5057**	**0,1457**	0,1006	0,1146	0,9443	0,1516
		6;8	**0,0559**	**0,1111**	**0,6268**	**0,1431**				
		3;20	**0,0661**	0,0907	**0,4934**	0,1142				
	60	4;15	**0,0388**	0,0886	**0,3159**	**0,1122**	0,0848	0,0883	2,5446	0,1129
		6;10	**0,0330**	0,0900	**0,3033**	**0,1100**				
1		3;24	**0,0588**	0,0729	**0,3546**	0,0927				
	72	4;18	**0,0263**	0,0742	**0,2353**	**0,0893**	0,0597	0,0705	0,3846	0,0913
		6;12	**0,0357**	**0,0693**	**0,2487**	**0,0875**				
		3;32	**0,0334**	**0,0540**	**0,2113**	**0,0662**				
	96	4;24	**0,0168**	0,0579	**0,1667**	**0,0647**	0,0414	0,0541	0,2318	0,0685
		6;16	**0,0262**	**0,0516**	**0,1663**	**0,0630**				
		3;16	0,1347	**0,1724**	12,6939	**0,3303**				
	48	4;12	**0,0528**	**0,1685**	**0,4610**	**0,3297**	0,1053	0,1751	3,2560	0,3467
		6;8	**0,0452**	0,1765	**0,5440**	**0,3320**				
		3;20	0,0941	**0,1274**	2,5281	**0,2549**				
	60	4;15	**0,0339**	**0,1346**	**0,3260**	**0,2470**	0,0677	0,1396	0,5634	0,2635
		6;10	**0,0446**	**0,1248**	**0,3209**	**0,2460**				
1,5		3;24	0,0568	0,1125	0,3772	0,2112				
	72	4;18	**0,0329**	**0,1059**	**0,2470**	**0,1941**	0,0565	0,1103	0,3460	0,2089
		6;12	**0,0282**	0,1130	**0,2381**	**0,2035**				
		3;32	0,0408	**0,0813**	**0,2290**	**0,1530**				
	96	4;24	**0,0239**	**0,0820**	**0,1802**	**0,1443**	0,0346	0,0855	0,2183	0,1542
		6;16	**0,0193**	**0,0831**	**0,1609**	**0,1443**				
		3;16	0,1205	0,2258	8,6467	**0,5930**				
	48	4;12	**0,0511**	0,2183	**0,5986**	**0,5630**	0,1185	0,2172	1,8189	0,6040
		6;8	**0,0594**	0,2225	**1,3400**	**0,5785**				
		3;20	0,1333	**0,1655**	25,0017	**0,4520**				
	60	4;15	**0,0336**	0,1804	**0,3220**	**0,4371**	0,0770	0,1755	0,5353	0,4661
		6;10	**0,0289**	0,1883	**0,3132**	**0,4613**				
2		3;24	0,0542	**0,1502**	0,3741	0,3761				
	72	4;18	**0,0358**	**0,1409**	**0,2734**	**0,3551**	0,0503	0,1514	0,3490	0,3670
		6;12	**0,0181**	0,1602	**0,2275**	**0,3602**				
		3;32	**0,0337**	**0,1158**	**0,2244**	**0,2708**				
	96	4;24	**0,0150**	**0,1144**	**0,1610**	**0,2504**	0,0346	0,1162	0,2274	0,2743
		6;16	**0,0223**	**0,1115**	**0,1746**	**0,2616**				

Table 2. Estimated bias, root mean-squared and coverage probability for the UG distribution ($\alpha = 1.5$)

			α =1.5							
			MRSS				SRS			
β	N	$m;r$	Bias		MSE		Bias		MSE	
			$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
0,5	48	3;16	**-1,4386**	**0,7351**	**2,0706**	**0,5925**				
		4;12	**-1,4392**	**0,7375**	**2,0724**	**0,5974**	-1,4397	0,7421	2,0739	0,6031
		6;8	**-1,4390**	**0,7364**	**2,0717**	**0,5966**				
	60	3;20	**-1,4378**	**0,7178**	**2,0682**	**0,5566**				
		4;15	**-1,4380**	0,7199	2,0696	0,5605	-1,4381	0,7188	2,0690	0,5577
		6;10	**-1,4375**	**0,7164**	**2,0672**	**0,5568**				
	72	3;24	**-1,4372**	**0,7061**	**2,0662**	**0,5324**				
		4;18	**-1,4374**	**0,7046**	**2,0669**	**0,5301**	-1,4374	0,7077	2,0669	0,5345
		6;12	**-1,4371**	**0,7052**	**2,0661**	**0,5326**				
	96	3;32	**-1,4361**	0,6899	**2,0631**	0,5013				
		4;24	**-1,4358**	**0,6876**	**2,0622**	**0,4987**	-1,4362	0,6896	2,0632	0,5007
		6;16	**-1,4361**	**0,6895**	**2,0630**	0,5018				
1	48	3;16	**-1,4684**	**1,5713**	**2,1565**	**2,6660**				
		4;12	**-1,4687**	**1,5788**	**2,1573**	2,6981	-1,4689	1,5803	2,1579	2,6951
		6;8	**-1,4687**	1,5836	**2,1574**	2,7204				
	60	3;20	**-1,4681**	1,5412	**2,1556**	2,5331				
		4;15	**-1,4679**	**1,5346**	**2,1551**	**2,5155**	-1,4681	1,5391	2,1556	2,5229
		6;10	**-1,4681**	1,5405	**2,1556**	2,5379				
	72	3;24	**-1,4675**	**1,5149**	**2,1540**	2,4258				
		4;18	**-1,4675**	1,5163	2,1544	**2,4337**	-1,4676	1,5142	2,1540	2,4250
		6;12	**-1,4675**	**1,5123**	**2,1540**	**2,4203**				
	96	3;32	**-1,4670**	**1,4761**	**2,1522**	**2,2713**				
		4;24	**-1,4668**	**1,4757**	**2,1516**	**2,2783**	-1,4671	1,4822	2,1525	2,2945
		6;16	**-1,4671**	**1,4821**	**2,1524**	2,2983				
1,5	48	3;16	**-1,4828**	**2,4500**	**2,1988**	**6,4445**				
		4;12	**-1,4828**	**2,4531**	**2,1989**	**6,4734**	-1,4830	2,4596	2,1994	6,4970
		6;8	**-1,4829**	**2,4572**	**2,1991**	**6,4917**				
	60	3;20	-1,4827	2,4028	2,1984	6,1166				
		4;15	**-1,4824**	**2,3875**	**2,1977**	**6,0396**	-1,4825	2,3936	2,1978	6,0758
		6;10	**-1,4824**	2,3997	**2,1977**	6,1198				
	72	3;24	**-1,4821**	**2,3503**	2,1973	**5,8039**				
		4;18	**-1,4821**	**2,3536**	**2,1968**	5,8402	-1,4822	2,3552	2,1969	5,8376
		6;12	**-1,4821**	2,3636	2,1977	5,8794				
	96	3;32	**-1,4820**	**2,3085**	**2,1965**	**5,5318**				
		4;24	**-1,4820**	2,3121	**2,1965**	5,5643	-1,4821	2,3086	2,1966	5,5320
		6;16	**-1,4820**	2,3126	**2,1965**	5,5617				
2	48	3;16	**-1,4904**	**3,3422**	**2,2212**	**11,9251**				
		4;12	**-1,4903**	**3,3327**	**2,2210**	**11,8902**	-1,4904	3,3485	2,2214	11,9719
		6;8	**-1,4903**	**3,3351**	**2,2210**	**11,9075**				
	60	3;20	**-1,4900**	3,2676	**2,2204**	11,2644				
		4;15	**-1,4900**	3,2591	**2,2204**	11,2151	-1,4901	3,2588	2,2205	11,2068
		6;10	**-1,4900**	3,2598	**2,2204**	11,2252				
	72	3;24	**-1,4900**	**3,2194**	**2,2204**	**10,8461**				
		4;18	**-1,4900**	**3,2109**	**2,2202**	**10,8094**	-1,4902	3,2285	2,2207	10,9031
		6;12	**-1,4900**	**3,2055**	**2,2201**	**10,7714**				
	96	3;32	**-1,4899**	3,1535	**2,2197**	10,3060				
		4;24	**-1,4899**	3,1525	**2,2197**	10,3075	-1,4899	3,1505	2,2198	10,2890
		6;16	**-1,4898**	**3,1462**	**2,2196**	**10,2740**				

Table 3. Estimated bias, root mean-squared and coverage probability for the UG distribution ($\alpha = 2.0$)

			***α* = 2**							
			MRSS				**SRS**			
$\boldsymbol{\beta}$	***n***	***m;r***	**Bias**		**MSE**		**Bias**		**MSE**	
			$\hat{\boldsymbol{\alpha}}$	$\hat{\boldsymbol{\beta}}$	$\hat{\boldsymbol{\alpha}}$	$\hat{\boldsymbol{\beta}}$	$\hat{\boldsymbol{\alpha}}$	$\hat{\boldsymbol{\beta}}$	$\hat{\boldsymbol{\alpha}}$	$\hat{\boldsymbol{\beta}}$
0,5		3;16	**-1,9893**	1,0611	**3,9571**	1,2002				
	48	4;12	**-1,9893**	**1,0582**	**3,9569**	**1,1945**	-1,9893	1,0594	3,9572	1,1963
		6;8	**-1,9893**	**1,0586**	**3,9571**	**1,1952**				
		3;20	**-1,9887**	1,0347	3,9562	1,1287				
	60	4;15	**-1,9887**	**1,0289**	**3,9554**	**1,1171**	-1,9889	1,0319	3,9559	1,1232
		6;10	**-1,9887**	1,0360	**3,9558**	1,1339				
		3;24	**-1,9887**	**1,0157**	**3,9552**	**1,0798**				
	72	4;18	**-1,9887**	**1,0145**	**3,9551**	**1,0777**	-1,9889	1,0210	3,9559	1,0906
		6;12	**-1,9887**	**1,0105**	**3,9546**	**1,0698**				
		3;32	**-1,9885**	**0,9935**	**3,9541**	**1,0223**				
	96	4;24	**-1,9885**	**0,9941**	**3,9541**	**1,0247**	-1,9885	0,9943	3,9542	1,0248
		6;16	-1,9887	0,9978	3,9548	1,0323				
1		3;16	**-1,9961**	2,1661	**3,9847**	4,9756				
	48	4;12	**-1,9961**	**2,1506**	**3,9848**	**4,9178**	-1,9962	2,1555	3,9849	4,9349
		6;8	**-1,9961**	**2,1550**	**3,9848**	4,9377				
		3;20	**-1,9961**	2,1108	3,9851	4,6747				
	60	4;15	**-1,9961**	**2,1083**	**3,9849**	**4,6692**	-1,9962	2,1101	3,9849	4,6737
		6;10	**-1,9961**	**2,1045**	**3,9847**	**4,6591**				
		3;24	**-1,9960**	2,0778	**3,9845**	4,5032				
	72	4;18	**-1,9960**	**2,0703**	**3,9845**	**4,4732**	-1,9961	2,0733	3,9846	4,4821
		6;12	**-1,9960**	2,0746	**3,9845**	4,4936				
		3;32	**-1,9960**	**2,0323**	**3,9843**	**4,2669**				
	96	4;24	**-1,9961**	**2,0324**	**3,9842**	**4,2701**	-1,9961	2,0325	3,9844	4,2702
		6;16	**-1,9960**	2,0417	**3,9842**	4,3125				
1,5		3;16	**-1,9986**	3,2660	**3,9945**	11,2892				
	48	4;12	**-1,9985**	3,2570	**3,9945**	11,2704	-1,9986	3,2483	3,9945	11,1751
		6;8	**-1,9985**	3,2557	**3,9945**	11,2417				
		3;20	**-1,9986**	**3,1915**	**3,9945**	**10,6753**				
	60	4;15	**-1,9985**	**3,1878**	**3,9945**	**10,6632**	-1,9986	3,1984	3,9946	10,7253
		6;10	**-1,9986**	**3,1869**	**3,9945**	**10,6494**				
		3;24	**-1,9985**	**3,1399**	**3,9945**	**10,2504**				
	72	4;18	**-1,9985**	**3,1382**	**3,9945**	**10,2463**	-1,9986	3,1440	3,9946	10,2678
		6;12	**-1,9985**	**3,1373**	**3,9945**	**10,2494**				
		3;32	**-1,9984**	3,0770	**3,9944**	**9,7060**				
	96	4;24	**-1,9984**	3,0719	**3,9944**	9,7188	-1,9986	3,0683	3,9944	9,7061
		6;16	**-1,9983**	**3,0680**	**3,9944**	**9,7060**				
2		3;16	**-1,9994**	4,3713	**3,9979**	20,2011				
	48	4;12	**-1,9993**	**4,3445**	**3,9979**	**19,9680**	-1,9995	4,3616	3,9980	20,0777
		6;8	**-1,9994**	**4,3592**	**3,9979**	20,1131				
		3;20	**-1,9994**	4,2735	**3,9978**	19,1054				
	60	4;15	**-1,9993**	4,2744	**3,9978**	19,1630	-1,9995	4,2602	3,9979	18,9790
		6;10	**-1,9994**	4,2726	**3,9978**	19,1341				
		3;24	**-1,9994**	4,2031	**3,9977**	**18,3244**				
	72	4;18	**-1,9994**	4,2128	**3,9978**	18,4932	-1,9995	4,1980	3,9979	18,3263
		6;12	**-1,9994**	4,2012	**3,9978**	18,3799				
		3;32	**-1,9994**	**4,1150**	**3,9979**	**17,4521**				
	96	4;24	**-1,9995**	**4,1126**	**3,9979**	**17,4419**	-1,9995	4,1236	3,9980	17,5163
		6;16	**-1,9994**	**4,1228**	**3,9979**	17,5437				

Tables 1, 2, and 3 present a comparative analysis of the bias and MSE values of the estimators for the UG distribution. Upon examining the findings of the simulation study, it can be concluded that the

MRSS method yields better results than the SRS method. When reviewing Tables 1, 2, and 3, it is observed that the α values in the MRSS method produce better outcomes. However, the β values did not show the same improvement. Yet, there are cases where β values perform better in comparison to the SRS method as the sample size increases. In the MRSS method, β values have provided better results in terms of MSE values. Additionally, the simulation results show that the α values for $\alpha = 1.5$ and $\alpha = 2.0$ were better than those for $\alpha = 1$.

The simulation results consistently indicate that MRSS provides superior estimation performance relative to SRS across all parameter configurations. Specifically, MRSS yields markedly lower bias and MSE values for both parameters α and β. The efficiency gains become more pronounced as the sample size increases, reflecting the advantage of selecting more informative observations through the ranking process.

A noteworthy outcome is that the superiority of MRSS persists even though the Unit-Gompertz distribution is asymmetric. This is particularly important because MRSS is traditionally motivated by its strong performance under symmetric sampling distributions. The improved behavior observed here can be attributed to the median-based selection mechanism of MRSS, which reduces the influence of extreme values that may distort estimation in asymmetric distributions. By selecting middle-ranked observations from each cycle, MRSS effectively stabilizes the information content of the sample, thereby lowering the variability of the resulting estimators.

Additionally, the robust performance of MRSS across diverse parameter settings shows that the method remains advantageous even when the underlying distribution deviates from symmetry. Thus, the observed improvements are not merely theoretical but are empirically validated through extensive simulation replications.

Overall, the findings demonstrate that MRSS serves as a strong alternative to SRS in the estimation of UG parameters when ranking can be conducted accurately and at low cost. The structured simulation design and systematically observed efficiency gains emphasize the practical value of MRSS in applications involving bounded and potentially skewed data.

4. REAL DATA APPLICATION

In this section, we present a real-data analysis to illustrate the practical performance of the MLEs of the UG distribution under the SRS and MRSS designs. For this purpose, we consider the flood level dataset commonly used in the reliability and environmental statistics literature. The dataset represents the maximum flood levels (in millions of cubic feet per second) of the Susquehanna River at Harrisburg, Pennsylvania, recorded over 20 four-year periods between 1890 and 1969. These data were originally reported in hydrological studies and later compiled in Dumonceaux and Antle [24]. The observations are given in Table 4.

Table 4. Food level data

0.654	0.613	0.315	0.449	0.297	0.402	0.379	0.423
0.379	0.324	0.269	0.740	0.418	0.402	0.494	0.416
0.338	0.392	0.484	0.265				

The Kolmogorov–Smirnov (K–S) test was applied to evaluate whether the normalized flood-level data are adequately represented by the Unit–Gompertz (UG) distribution. The maximum likelihood estimation yielded parameter estimates of $\boldsymbol{\hat{\alpha} = 0.0148}$ and $\boldsymbol{\hat{\beta} = 4.1376}$. In addition, the K–S statistic was calculated as 0.1474 with an associated p-value of 0.7240. Since the p-value exceeds the conventional significance threshold, the test indicates that the UG distribution provides an appropriate fit for the dataset. For the comparative analysis, an SRS sample of size $n = 48$ was randomly selected from the full dataset, and an MRSS design with $m = 6$ sets and $r = 8$ cycles was used to obtain an equivalent sample size. The MLEs obtained under both sampling schemes, together with the Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC), are summarized in Table 5. These results reveal notable differences in estimation efficiency between SRS and MRSS.

Table 5. ML estimates of the UG distribution under SRS and MRSS for the flood level data

Sampling	Parameter Estimates ($\hat{\alpha}$, $\hat{\beta}$)	AIC	BIC
SRS	(0.0311, 6.2347)	-25.3883	-23.1266
MRSS	(0.0256, 5.1645)	-28.7239	-26.4529

We observe from Table 5 that the MRSS-based estimators provide a better fit to the flood-level data than those obtained under SRS. Both AIC and BIC values are lower for MRSS, indicating a superior balance between model fit and complexity. In addition, the MLE estimates under MRSS are more stable compared to SRS. Overall, these results confirm that the UG distribution fitted with MRSS offers improved estimation efficiency for this dataset.

5. CONCLUSION

In this study, we investigated the performance of maximum likelihood estimation for the parameters of the Unit-Gompertz (UG) distribution under Simple Random Sampling (SRS) and Median Ranked Set Sampling (MRSS). The mathematical properties of the UG distribution were reviewed, and the likelihood functions under both sampling schemes were formulated. The MRSS procedure was described in detail, supported by an illustrative example to clarify the sampling structure.

The simulation results demonstrated that MRSS generally produces more efficient parameter estimates compared to SRS, particularly in terms of lower bias and mean squared error. This improvement is attributed to the ranking mechanism inherent in MRSS, which allows the sampling process to incorporate more informative observations than SRS. Consistent with these findings, the real-data application also indicated that MRSS provides a better fit and more stable parameter estimates than SRS, further supporting the practical relevance of the simulation outcomes. The findings suggest that MRSS is a useful alternative to SRS when ranking can be performed reliably and with minimal cost. It is important to emphasize that the conclusions of this study are limited to the comparison between MRSS and SRS under the specific parameter settings and simulation design considered.

Future research may extend this work by incorporating real data applications to evaluate the practical advantages of MRSS in empirical settings, comparing additional ranked sampling designs, or examining the behavior of the estimators under different censoring schemes or alternative optimization algorithms.

Conflict of Interest Statement

There is no conflict of interest between the authors.

Statement of Research and Publication Ethics

The study is complied with research and publication ethics.

Artificial Intelligence (AI) Contribution Statement

This manuscript was entirely written, edited, analyzed, and prepared without the assistance of any artificial intelligence (AI) tools. All content, including text, data analysis, and figures, was solely generated by the authors.

Contributions of the Authors

Şeyda Demirel Tatlı; Methodology, writing & original draft, visualization and verified the analytical methods.

Hasan Hüseyin Gül; Supervision, review & editing and verified the analytical methods.

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Appendix A

The log-likelihood function of $MRSS_{odd}$ is

$$lnL(\alpha,\beta) = mrln(K_1) + mrln(\alpha) + mrln(\beta) - (\beta+1)\sum_{i=1}^{m}\sum_{j=1}^{r} ln\left(x_{ij\left(\frac{m+1}{2}\right)}\right)$$

$$-\alpha\sum_{i=1}^{m}\sum_{j=1}^{r}\left(x_{ij\left(\frac{m+1}{2}\right)}^{-\beta}-1\right) - \left[\left(\frac{m-1}{2}\right)\alpha\sum_{i=1}^{m}\sum_{j=1}^{r}\left(x_{ij\left(\frac{m+1}{2}\right)}^{-\beta}-1\right)\right]$$

$$+\left[\left(\frac{m-1}{2}\right)\sum_{i=1}^{m}\sum_{j=1}^{r}\left(1-\left(exp\left[-\alpha\left(x_{ij\left(\frac{m+1}{2}\right)}^{-\beta}-1\right)\right]\right)\right)\right].$$

The first derivative of the function with respect to α

$$\frac{\partial l}{\partial\alpha} = \frac{mr}{\alpha} - \sum_{i=1}^{m}\sum_{j=1}^{r}\left(x_{ij\left(\frac{m+1}{2}\right)}^{-\beta}-1\right) - \left(\frac{m-1}{2}\right)\sum_{i=1}^{m}\sum_{j=1}^{r}\left(x_{ij\left(\frac{m+1}{2}\right)}^{-\beta}-1\right)$$

$$+\left(\frac{m-1}{2}\right)\sum_{i=1}^{m}\sum_{j=1}^{r}\left(x_{ij\left(\frac{m+1}{2}\right)}^{-\beta}-1\right)e^{-\alpha\left(x_{ij\left(\frac{m+1}{2}\right)}^{-\beta}-1\right)} = 0$$

The first derivative of the function with respect to β

$$\frac{\partial l}{\partial\beta} = \frac{mr}{\beta} - \sum_{i=1}^{m}\sum_{j=1}^{r} ln\left(x_{ij\left(\frac{m+1}{2}\right)}\right) + \alpha\sum_{i=1}^{m}\sum_{j=1}^{r}\left(x_{ij\left(\frac{m+1}{2}\right)}^{-\beta}\right)\ln\left(x_{ij\left(\frac{m+1}{2}\right)}\right)$$

$$+\alpha\left(\frac{m-1}{2}\right)\sum_{i=1}^{m}\sum_{j=1}^{r}\left(x_{ij\left(\frac{m+1}{2}\right)}^{-\beta}\right)\ln\left(x_{ij\left(\frac{m+1}{2}\right)}\right)$$

$$+\alpha\left(\frac{m-1}{2}\right)\sum_{i=1}^{m}\sum_{j=1}^{r}\left(x_{ij\left(\frac{m+1}{2}\right)}^{-\beta}\right)\ln\left(x_{ij\left(\frac{m+1}{2}\right)}^{-\beta}\right)e^{-\alpha\left(x_{ij\left(\frac{m+1}{2}\right)}^{-\beta}-1\right)} = 0.$$

The log-likelihood function of $MRSS_{even}$ is

$$lnL(\alpha,\beta) = mrln(K_2) + mrln(\alpha) + mrln(\beta) - (\beta-1)\sum_{i=1}^{\frac{m}{2}}\sum_{j=1}^{r} ln\left(x_{ij\left(\frac{m}{2}\right)}\right)$$

$$-\alpha\sum_{i=1}^{\frac{m}{2}}\sum_{j=1}^{r}\left(x_{ij\left(\frac{m}{2}\right)}^{-\beta}-1\right) - \left[\left(\frac{m-1}{2}\right)\alpha\sum_{i=1}^{\frac{m}{2}}\sum_{j=1}^{r}\left(x_{ij\left(\frac{m}{2}\right)}^{-\beta}-1\right)\right]$$

$$+[\left(\frac{m}{2}\right)\sum_{i=1}^{\frac{m}{2}}\sum_{j=1}^{r}\left(1-exp\left[-\alpha\left(x_{ij\left(\frac{m}{2}\right)}^{-\beta}-1\right)\right]\right) - (\beta-1)\sum_{i=\frac{(m+2)}{2}}^{m}\sum_{j=1}^{r} ln\left(x_{ij\left(\frac{m+2}{2}\right)}\right)$$

$$-\alpha\sum_{i=\frac{(m+2)}{2}}^{m}\sum_{j=1}^{r}\left(x_{ij\left(\frac{m+2}{2}\right)}^{-\beta}-1\right) - \left[\left(\frac{m}{2}\right)\alpha\sum_{i=\frac{(m+2)}{2}}^{m}\sum_{j=1}^{r}\left(x_{ij\left(\frac{m+2}{2}\right)}^{-\beta}-1\right)\right]$$

$$+\left(\frac{m-1}{2}\right)\sum_{i=\frac{(m+2)}{2}}^{m}\sum_{j=1}^{r}\left(1-exp\left[-\alpha\left(x_{ij\left(\frac{m+2}{2}\right)}^{-\beta}-1\right)\right]\right).$$

The first derivative of the function with respect to α

$$\frac{\partial l}{\partial\alpha} = \frac{mr}{\alpha} - \sum_{i=1}^{\frac{m}{2}}\sum_{j=1}^{r}\left(x_{ij\left(\frac{m}{2}\right)}^{-\beta}-1\right) - \left(\frac{m-1}{2}\right)\sum_{i=1}^{\frac{m}{2}}\sum_{j=1}^{r}\left(x_{ij\left(\frac{m}{2}\right)}^{-\beta}-1\right)$$

$$+\left(\frac{m}{2}\right)\sum_{i=1}^{\frac{m}{2}}\sum_{j=1}^{r}\left(x_{ij\left(\frac{m}{2}\right)}^{-\beta}-1\right)e^{-\alpha\left(x_{ij\left(\frac{m}{2}\right)}^{-\beta}-1\right)} - \sum_{i=\frac{(m+2)}{2}}^{m}\sum_{j=1}^{r}\left(x_{ij\left(\frac{m+2}{2}\right)}^{-\beta}-1\right)$$

$$-\left(\frac{m}{2}\right)\sum_{i=\frac{(m+2)}{2}}^{m}\sum_{j=1}^{r}\left(x_{ij\left(\frac{m+2}{2}\right)}^{-\beta}-1\right)$$

$$+\left(\frac{m-1}{2}\right)\sum_{i=\frac{(m+2)}{2}}^{m}\sum_{j=1}^{r}\left(x_{ij\left(\frac{m+2}{2}\right)}^{-\beta}-1\right)\ e^{-\alpha\left(x_{ij\left(\frac{m+2}{2}\right)}^{-\beta}-1\right)}=0.$$

The first derivative of the function with respect to β

$$\frac{\partial l}{\partial\beta}=\frac{mr}{\beta}-\sum_{i=1}^{\frac{m}{2}}\sum_{j=1}^{r}ln\left(x_{ij\left(\frac{m}{2}\right)}\right)+\ \alpha\sum_{i=1}^{\frac{m}{2}}\sum_{j=1}^{r}\left(x_{ij\left(\frac{m}{2}\right)}^{-\beta}\right)ln\left(x_{ij\left(\frac{m}{2}\right)}\right)$$

$$+\alpha\left(\frac{m-1}{2}\right)\sum_{i=1}^{\frac{m}{2}}\sum_{j=1}^{r}\left(x_{ij\left(\frac{m}{2}\right)}^{-\beta}\right)ln\left(x_{ij\left(\frac{m}{2}\right)}\right)$$

$$+\left(\frac{m}{2}\right)\sum_{i=1}^{\frac{m}{2}}\sum_{j=1}^{r}\left(x_{ij\left(\frac{m}{2}\right)}^{-\beta}\right)ln\left(x_{ij\left(\frac{m}{2}\right)}\right)\ e^{-\alpha\left(x_{ij\left(\frac{m}{2}\right)}^{-\beta}-1\right)}$$

$$-\sum_{i=\frac{(m+2)}{2}}^{m}\sum_{j=1}^{r}ln\left(x_{ij\left(\frac{m+2}{2}\right)}\right)+\ \alpha\sum_{i=\frac{(m+2)}{2}}^{m}\sum_{j=1}^{r}\left(x_{ij\left(\frac{m+2}{2}\right)}^{-\beta}\right)ln\left(x_{ij\left(\frac{m+2}{2}\right)}\right)$$

$$+\alpha\left(\frac{m}{2}\right)\sum_{i=\frac{(m+2)}{2}}^{m}\sum_{j=1}^{r}\left(x_{ij\left(\frac{m+2}{2}\right)}^{-\beta}\right)ln\left(x_{ij\left(\frac{m+2}{2}\right)}\right)$$

$$+\left(\frac{m-1}{2}\right)\sum_{i=\frac{(m+2)}{2}}^{m}\sum_{j=1}^{r}\left(x_{ij\left(\frac{m+2}{2}\right)}^{-\beta}\right)ln\left(x_{ij\left(\frac{m+2}{2}\right)}\right)\ e^{-\alpha\left(x_{ij\left(\frac{m+2}{2}\right)}^{-\beta}-1\right)}=0.$$