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BOUNDEDNESS OF THE AMBIGUITY FUNCTION ON HARDY AND BMO SPACES

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ABSTRACT

This paper demonstrates that the ambiguity function, which is well-known for its significant engineering applications, provides results on boundedness on the Hardy and BMO (Bounded Mean Oscillation) spaces, which are also of central importance in various mathematical fields of study, including Fourier and harmonic analysis. Furthermore, the study meticulously investigates the Hardy and BMO distances between two ambiguities that depend on different windows and signals.

1. INTRODUCTION

Let h denote a function on $\mathbb{R}^d$. The modulation operator of h is specified as $M_\mu h(y) = h(y)\exp(2\pi i\mu \cdot y)$ for $\mu, y \in \mathbb{R}^d$, while the translation operator is on $L_u h(y) = h(y-u)$ for $y \in \mathbb{R}^d$. L and M are sometimes known as the time and frequency shift operators, respectively. Operators $L_u M_\mu$ or $M_\mu L_u$ are known as time frequency shifts. L and M do not commute. However, we observe instantly the canonical commutation relations

$$M_\mu L_u = e^{2\pi i u\cdot\mu} L_u M_\mu.$$

It is evident that L and M commute if and only if $u \cdot \mu \in \mathbb{Z}$.

If $p \in [1,\infty)$, the Lebesgue spaces which is denoted by $L^p(\mathbb{R}^d)$, is defined as the set of complex-valued measurable functions on $\mathbb{R}^d$ that satisfy

$$\int |h(y)|^p dy < \infty.$$

If $h \in L^p(\mathbb{R}^d)$, the L^p norm of h is defined by

$$\|h\|_{L^p} = \|h\|_p = \left(\int |h(y)|^p dy\right)^{1/p} < \infty.$$

$L^p(\mathbb{R}^d)$ is a Banach space with the norm $\|\cdot\|_p$.

For $p = \infty$, the L^∞ norm is the essential supremum of the function's absolute value:

$$\|h\|_\infty = ess \sup_{x\in\mathbb{R}^d} |h(x)|\,.$$

The $L^\infty(\mathbb{R}^d)$ space is the set of all measurable functions that are essentially bounded on $\mathbb{R}^d$.

It is known that a $\mathbb{C}$ valued function h defined on $\mathbb{R}^d$ is called a locally integrable function if $\int |h(x)|dx < \infty$, where $K \subset \mathbb{R}^d$ is compact and the integral is over K. $L^1_{loc}(\mathbb{R}^d)$ denotes the spaces of locally integrable functions.

Let $h \in L^1(\mathbb{R}^d)$. Let us define $\hat{h}$ (or $\mathcal{F}h$) by

$$\mathcal{F}h(z) = \hat{h}(z) = \int h(x) e^{-2\pi i x\cdot z}\, dx, \quad z \in \mathbb{R}^d,$$

where $x \cdot z = \sum_{i=1}^{d} x_i z_i$ is the usual scalar product on $\mathbb{R}^d$. $\hat{h}$ is referred to as the Fourier transform of h.

The following definition constitutes the cross-Wigner-Ville distribution of functionals h and g, which are elements of the $L^2(\mathbb{R}^d)$ space:

$$W(h, g)(x, w) = \int e^{-2\pi i t\cdot w} h(x + t/2)\, \overline{g(x - t/2)} dt.$$

If we write h instead of g, then $W(h, h) = Wh$ is known as the Wigner distribution of h. In the context of analyzing non-stationary signals, it is imperative to employ both time and frequency representations, as the Fourier analysis, a valuable instrument for the study of stationary signals, is inadequate for the comprehensive analysis of non-stationary signals. The Wigner distribution is the most often used time-frequency representation because it offers a high-resolution representation in both time and frequency for non-stationary signals. However, in 1950s, the theory of the Wigner distribution underwent a significant reformation within the domain of sonar and radar signal analysis. The fundamental premise of this theory hinges upon the utilization of the echo from a transmitted signal to determine the position and velocity of a target. In [11], Woodward's seminal contribution to the field of mathematical analysis was the introduction of a novel function for the analysis of sonar and radar signals which was named the "ambiguity function."

The cross- ambiguity-function of functionals h and g, which are elements of the $L^2(\mathbb{R}^d)$ space, is defined as follows:

$$A(h, g)(x, w) = \int e^{-2\pi i t\cdot w} h(t + x/2)\, \overline{g(t - x/2)} dt,$$

if h is equal to g, then $Ah(x, w)$ is referred to the autoambiguity function of h is

$$Ah(x, w) = \int e^{-2\pi i t\cdot w} h(t + x/2)\, \overline{h(t - x/2)} dt.$$

It is important to note that the term "ambiguity functions" is used to refer to both autoambiguity and cross ambiguity functions, [2,7-9,11].

There are significant connections between the theory of Hardy Spaces (HS) and many areas of mathematical study, such as Fourier analysis, harmonic analysis, operator theory and singular integrals, signal and image processing, and control theory. The literature has demonstrated that the HS is more appropriate than the Lebesgue space for certain harmonic analysis concerns. The maximal functions can be defined as follows, and this will allow us to give an equivalent definition of HS $H^1(\mathbb{R}^d)$: We are going to take a function that is both integrable and smooth. This function will be denoted by ' φ ' and its domain will be the d-dimensional Euclid space, with its support lying in the unit ball. In addition, it should be $\int_{\mathbb{R}^d} \varphi = 1$. Let us set $\varphi_t(y) = 1/t^d\, \varphi(y/t),\ t > 0$. For an integrable function h, the maximal operator, represented by the symbol $\mathcal{M}_\varphi$, is defined as follows:

$$\mathcal{M}_\varphi h(y) = \sup_{t>0} |h * \varphi_t(y)|.$$

The HS $H^1(\mathbb{R}^d)$ represents the linear space of all $h \in L^1(\mathbb{R}^d)$ if, for some $\varphi \in \mathcal{S}(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \varphi = 1$, $\mathcal{M}_\varphi h$ is in $L^1(\mathbb{R}^d)$, where $\mathcal{S}(\mathbb{R}^d)$ is the Schwartz space. If h belongs to HS, then L_u is in HS and $\|L_u h\|_{H^1} = \|h\|_{H^1}$.

The space of functions of Bounded Mean Oscillation (BMO) provides a framework for describing functions whose local fluctuations are uniformly controlled. While a BMO function is not necessarily bounded, its oscillation, quantified by the mean difference between the function and its average value, remains limited across all local domains. The class of functions whose deviation from their means over cubes is bounded is the space of functions of bounded mean oscillation which constitutes crucial function space in harmonic analysis, or BMO (also called the John-Nirenberg space). In [6], John-Nirenberg developed the space $BMO(\mathbb{R}^d)$ of functions with bounded mean oscillation. $BMO(\mathbb{R}^d)$ represents the space of all functions $h \in L^1_{loc}(\mathbb{R}^d)$ such that

$$\|h\|_{BMO} = \sup_{Q\subset\mathbb{R}^d} |Q|^{-1} \int |h(x) - Q(h)|\ dx < \infty,$$

where the integral is over Q and the supremum is taken over the balls Q in $\mathbb{R}^d$ of measure $|Q|$, and $Q(h)$ stands for the mean of h on Q, namely,

$$Q(h) = |Q|^{-1} \int h(x)\, dx \le |Q|^{-1} \int |h(x)|\, dx \le M < \infty. \tag{1}$$

The space known as $BMO(\mathbb{R}^d)$ is the dual of HS. Functionally, BMO occupies an intermediary role, situated between the space of essentially bounded functions, L^∞ , and the Lebesgue spaces, L^p (for $p < \infty$). This position underscores its fundamental importance across diverse fields, including the study of Partial Differential Equations, the analysis of singular integral operators, and its intimate connection with Hardy spaces, thereby cementing BMO spaces as an essential topic in theoretical mathematics, [1,3-5,7,10].

2. MAIN RESULTS

2.1. Boundedness on Hardy Space

The HS-boundedness of the ambiguity function will be our first topic.

Lemma 1. If $h \in L^1(\mathbb{R}^d)$, $g \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, then $A(h,g)(\cdot,w) \in L^1(\mathbb{R}^d)$.

Proof. For a fixed w in $\mathbb{R}^d$, the function $A(h,g)(x,w)$ depends on x. By changing variable $t - x/2 = a$, we write

$$\begin{aligned} A(h,g)(x,w) &= \int e^{-2\pi i t\cdot w} h(t + x/2)\, \overline{g(t - x/2)} dt \\ &= \int e^{-2\pi i a\cdot w - \pi i x\cdot w} h(a + x)\, \overline{g(a)}\, da. \end{aligned} \tag{2}$$

Also by using Fubini's Theorem, we get

$$\begin{aligned} \|A(h,g)(\cdot,w)\|_1 &= \int \left| \int e^{-2\pi i a\cdot w - \pi i x\cdot w} h(a + x)\, \overline{g(a)}\, da \right| dx \\ &\le \int |g(a)| \left(\int |h(a + x)|\, dx \right) da \\ &= \|h\|_1 \int |g(a)|\, da \ = \|h\|_1\, \|g\|_1. \end{aligned}$$

Consequently, $A(h,g)(\cdot,w) \in L^1(\mathbb{R}^d)$.

Theorem 1. Let g be in $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. The function $(\cdot, g): H^1(\mathbb{R}^d) \to H^1(\mathbb{R}^d)$, $h \to A(h, g)(\cdot, w)$, is bounded. Moreover,

$$\|A(h,g)(\cdot,w)\|_{H^1} \leq \|h\|_{H^1} \|g\|_1.$$

Proof. By using the equality (2), we get

$$\begin{aligned}(A(h,g)(\cdot,w) * \varphi_t)(x) &= \int A(h,g)(x-y,w)\,\varphi_t(y)\,dy \\ &= \int \left(\int e^{-2\pi i\left(a+\frac{x-y}{2}\right)\cdot w} h(a+x-y)\,\overline{g(a)}\,da \right) \varphi_t(y)\,dy \\ &= \int \left(\int e^{-\pi i(x-y)\cdot w} L_{-a}h(x-y)\,\varphi_t(y)\,dy \right) \overline{g(a)}\, e^{-2\pi i a\cdot w}\,da.\end{aligned}$$

Applying the Minkowski's inequality for integrals and by using the translation invariant property of Hardy space, we have

$$\begin{aligned}\|A(h,g)(\cdot,w)\|_{H^1} &= \int \sup_{t>0} |(A(h,g)(\cdot,w) * \varphi_t)(x)|\,dx \\ &\leq \int \left|\overline{M_w g(a)}\right| \left(\int \sup_{t>0} \left(\int \left|e^{-\pi i(x-y)\cdot w}\right| |L_{-a}h(x-y)|\,|\varphi_t(y)|\,dy \right) dx \right) da \\ &= \int |g(a)| \left(\int \sup_{t>0} (|L_{-a}h| * |\varphi_t|)(x)\,dx \right) da \\ &= \int |g(a)|\, \|L_{-a}h\|_{H^1}\,da \\ &= \|h\|_{H^1} \int |g(a)|\,da = \|h\|_{H^1} \|g\|_1.\end{aligned}$$

This concludes the proof.

Theorem 2. Let $\phi,\ \psi \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. If $h_1,\ h_2 \in H^1(\mathbb{R}^d)$, then

$$\|A(h_1,\phi)(\cdot,w) - A(h_2,\psi)(\cdot,w)\|_{H^1} \leq \|\phi-\psi\|_1 \|h_1\|_{H^1} + \|\psi\|_1 \|h_1 - h_2\|_{H^1}.$$

Proof. We write

$$\begin{aligned}&\|A(h_1,\phi)(\cdot,w) - A(h_2,\psi)(\cdot,w)\|_{H^1} \\ &\leq \|A(h_1,\phi)(\cdot,w) - A(h_1,\psi)(\cdot,w)\|_{H^1} + \|A(h_1,\psi)(\cdot,w) - A(h_2,\psi)(\cdot,w)\|_{H^1}.\end{aligned}$$

With $A(h_1,\psi)(x,w) - A(h_2,\psi)(x,w) = A(h_1 - h_2,\psi)(x,w)$, we obtain

$$\Big(\big(A(h_1,\psi)(\cdot,w) - A(h_2,\psi)(\cdot,w)\big) * \varphi_t(\cdot)\Big)(x) = \int \overline{M_w g(a)}\,\big(M_{-w/2}L_{-a}(h_1 - h_2) * \varphi_t\big)(x)\,da.$$

Then by Theorem 1, we get

$$\|A(h_1,\psi)(\cdot,w) - A(h_2,\psi)(\cdot,w)\|_{H^1} \leq \|\psi\|_1 \|h_1 - h_2\|_{H^1}. \tag{3}$$

Moreover, it is evident that

$$A(h_1,\phi)(x,w) - A(h_1,\psi)(x,w) = \int e^{-2\pi i a\cdot w - \pi i x\cdot w} h_1(a+x)\,\overline{(\phi-\psi)(a)}\,da$$

and

$$\Big(\big(A(h_1,\phi) - A(h_1,\psi)\big)(\cdot,w) * \varphi_t(\cdot)\Big)(x) = \int \overline{M_w(\phi-\psi)(a)}\,\big(M_{-w/2}L_{-a}h_1 * \varphi_t\big)(x)\,da.$$

Again by Theorem 1, we write

$$\|A(h_1,\phi)(\cdot,w) - A(h_1,\psi)(\cdot,w)\|_{H^1} \leq \|\phi - \psi\|_1 \|h_1\|_{H^1}. \tag{4}$$

Then from the equations (3) and (4), we obtain

$$\|A(h_1,\phi)(\cdot,w) - A(h_2,\psi)(\cdot,w)\|_{H^1} \leq \|\phi - \psi\|_1 \|h_1\|_{H^1} + \|\psi\|_1 \|h_1 - h_2\|_{H^1}.$$

2.2. Boundedness on BMO Space

Now, we will investigate the BMO- boundedness of ambiguity function. In order to proceed, the following lemma must be established.

Lemma 2. Let us assume that g is a function belonging to the $L^1(\mathbb{R}^d)$ and is compactly supported (cs). If $h \in L^1_{loc}(\mathbb{R}^d)$, then $A(h,g)(\cdot,w) \in L^1_{loc}(\mathbb{R}^d)$.

Proof. Since $A(h,g)(x,w)$ is a function of the first variable and

$$|A(h,g)(x,w)| \leq \int |g(a)||h(a+x)|da,$$

we can get for any compact ball $B \subset \mathbb{R}^d$

$$\int_B |A(h,g)(x,w)|\, dx \leq \int_{\mathbb{R}^d} |g(a)| \left(\int_B |h(a+x)|dx \right) da.$$

Let $K \subset a + B$. As $K \subset suppg + B$, where $suppg$ is the closure of the set $\{x \in \mathbb{R}^d \big| g(x) \neq 0\}$, is a closed and bounded set in $\mathbb{R}^d$ and $h \in L^1_{loc}(\mathbb{R}^d)$, hence we get

$$\int_B |A(h,g)(x,w)|\, dx \leq \int_{\mathbb{R}^d} |g(a)| \left(\int_K |h(v)|dv \right) da = N\, \|g\|_1.$$

So, $A(h,g)(\cdot,w)$ is a locally integrable function.

Theorem 3. Assume that $g \in L^1(\mathbb{R}^d)$ is a function whose closed support is a compact set. The operator $A(\cdot,g): BMO(\mathbb{R}^d) \to BMO(\mathbb{R}^d)$, $h \to A(h,g)(\cdot,w)$ is bounded. Moreover,

$$\|A(h,g)(\cdot,w)\|_{BMO} \leq (\|h\|_{BMO} + 2M)\, \|g\|_1.$$

Proof. Let $Q \subset \mathbb{R}^d$ be an arbitrary ball and $h \in BMO(\mathbb{R}^d)$. Then $h \in L^1_{loc}(\mathbb{R}^d)$ and so $A(h,g) \in L^1_{loc}(\mathbb{R}^d)$ by Lemma 2. By using the Fubini's Theorem, we have

$$Q\big(A(h,g)\big) = |Q|^{-1} \int_Q A(h,g)(z,w)\, dz$$

$$= \int_{\mathbb{R}^d} \overline{e^{2\pi\imath a\cdot w} g(a)} \left(|Q|^{-1} \int_Q h(a+z) e^{-\pi i z\cdot w}\, dz \right) da$$

and from here, we write

$$\|A(h,g)(\cdot,w)\|_{BMO} = \sup_{Q\subset\mathbb{R}^d} |Q|^{-1} \int_Q \big|A(h,g)(x,w) - Q\big(A(h,g)\big)\big|\, dx$$

$$= \sup_{Q\subset\mathbb{R}^d} |Q|^{-1} \int_Q \left| \int_{\mathbb{R}^d} \overline{e^{2\pi\imath a\cdot w} g(a)} \left(h(a+x)e^{-\pi i x\cdot w} - |Q|^{-1} \int_Q h(a+z) e^{-\pi i z\cdot w}\, dz \right) da \right| dx$$

$$\leq \int_{\mathbb{R}^d} |g(a)| \left(\sup_{Q\subset\mathbb{R}^d} |Q|^{-1} \int_Q \left| h(a+x)e^{-\pi i x\cdot w} - |Q|^{-1} \int_Q h(a+z) e^{-\pi i z\cdot w}\, dz \right| dx \right) da$$

$$\leq \int_{\mathbb{R}^d} |g(a)| \left(\sup_{Q\subset\mathbb{R}^d} |Q|^{-1} \int_Q \left| h(a+x)e^{-\pi i x\cdot w} - |Q|^{-1} e^{-\pi i x\cdot w} \int_Q h(a+z)\,dz \right| dx \right.$$
$$+ \sup_{Q\subset\mathbb{R}^d} |Q|^{-1} \int_Q \left| |Q|^{-1} e^{-\pi i x\cdot w} \int_Q h(a+z)\,dz \right| dx$$
$$\left. + \sup_{Q\subset\mathbb{R}^d} |Q|^{-1} \int_Q \left| |Q|^{-1} \int_Q h(a+z) e^{-\pi i z\cdot w}\,dz \right| dx \right) da$$
$$\leq \int_{\mathbb{R}^d} |g(a)| \left(\sup_{Q\subset\mathbb{R}^d} |Q|^{-1} \int_Q \left| h(a+x) - |Q|^{-1} \int_Q h(a+z)\,dz \right| dx \right.$$
$$\left. + 2 \sup_{Q\subset\mathbb{R}^d} |Q|^{-1} \int_Q \left| |Q|^{-1} \int_Q h(a+z)\,dz \right| dx \right) da.$$

If we say $U = Q + a$, $a \in \mathbb{R}^d$, and using the inequality (1), we get

$$\|A(h,g)(\cdot,w)\|_{BMO}$$
$$\leq \int_{\mathbb{R}^d} |g(a)| \left(\sup_{U\subset\mathbb{R}^d} |U|^{-1} \int_U \left| h(u) - |U|^{-1} \int_U h(v)\,dv \right| du \right.$$
$$\left. + 2 \sup_{U\subset\mathbb{R}^d} |U|^{-1} \int_U \left| |U|^{-1} \int_U h(v)\,dv \right| du \right) da$$
$$\leq \int_{\mathbb{R}^d} |g(a)| (\|h\|_{BMO} + 2|U|^{-1} M |U|) da$$
$$= (\|h\|_{BMO} + 2M) \int_{\mathbb{R}^d} |g(a)| da$$
$$= (\|h\|_{BMO} + 2M)\ \|g\|_1.$$

Theorem 4. Let $\phi,\ \psi \in L^1(\mathbb{R}^d)$ be two compactly supported functions. If $h_1,\ h_2 \in BMO(\mathbb{R}^d)$, then we have

$$\|A(h_1,\psi)(\cdot,w) - A(h_2,\phi)(\cdot,w)\|_{BMO} \leq \|\psi - \phi\|_1 (\|h_1\|_{BMO} + 2M) + \|\phi\|_1 (\|h_1 - h_2\|_{BMO} + 2M).$$

Proof. Let $\phi,\ \psi \in L^1(\mathbb{R}^d)$ be two cs functions and $h_1,\ h_2 \in BMO(\mathbb{R}^d)$. So, $h_1,\ h_2 \in L^1_{loc}(\mathbb{R}^d)$ and so $A(h_1,\psi)(\cdot,w),\ A(h_2,\phi)(\cdot,w) \in L^1_{loc}(\mathbb{R}^d)$ by Lemma 2. Moreover, since $A(h_1,\psi)(x,w) - A(h_1,\phi)(x,w) = A(h_1,\psi - \phi)(x,w)$ and $A(h_1,\phi)(x,w) - A(h_2,\phi)(x,w) = A(h_1 - h_2,\phi)(x,w)$, we obtain by Theorem 3,

$$\|A(h_1,\psi)(\cdot,w) - A(h_2,\phi)(\cdot,w)\|_{BMO}$$
$$\leq \|A(h_1,\psi)(\cdot,w) - A(h_1,\phi)(\cdot,w)\|_{BMO} + \|A(h_1,\phi)(\cdot,w) - A(h_2,\phi)(\cdot,w)\|_{BMO}$$
$$= \|A(h_1,\psi - \phi)(\cdot,w)\|_{BMO} + \|A(h_1 - h_2,\phi)(\cdot,w)\|_{BMO}$$
$$\leq \|\psi - \phi\|_1 (\|h_1\|_{BMO} + 2M) + \|\phi\|_1 (\|h_1 - h_2\|_{BMO} + 2M).$$

3. RESULTS AND DISCUSSION

In present work, we have researched the ambiguity function which is defined as the two-dimensional autocorrelation of a waveform in both time and frequency and represents the limitations and utility of different waveforms and provides guidance for selecting appropriate waveforms for various applications. We investigate the boundedness of the ambiguity function on $H^1(\mathbb{R}^d)$ and $BMO(\mathbb{R}^d)$. Mathematically, this study on the ambiguity function has focused on challenging subjects like Hardy and BMO spaces, which play vital roles in theoretical mathematics within the context of operator theory and singular integrals. This rich theoretical background is expected to serve as a bridge with the potential to yield novel and applicable results in engineering.

Statement of Research and Publication Ethics

The study is complied with research and publication ethics.

Artificial Intelligence (AI) Contribution Statement

This manuscript was entirely written, edited, analyzed, and prepared without the assistance of any artificial intelligence (AI) tools. All content, including text, data analysis, and figures, was solely generated by the author.

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